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Relatively Free Semigroups of Intermediate Growth

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The problem of calculating the growth of a finitely generated (f.g.) semigroup satisfying the given system of identities is considered. Examples of relatively free semigroups having intermediate growth, new growth criteria, and constructions are given. The main ideas employed in the proofs are based on some new points of view on growth. Our method is also useful for constructing various examples of f.g. nilpotent and nil-semigroups having intermediate growth and arbitrarily small Gelfand–Kirillov superdimension.

CONTENTS

1. INTRODUCTION

The notion of the growth function for groups was introduced by Milnor [17] in 1968. Connections between identities and growth functions for various algebras have been much studied (see the monographs [13, 34]).

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Moreover, several classic results about growth can be viewed as results of relationships between the growth and identities for certain algebraic systems.

In 1957 Shirshov [26] in fact proved that every finitely generated (f.g.) associative algebra over a field satisfying a nontrivial polynomial identity has polynomial growth. In 1968 Milnor [18] and Wolf [35] proved that every f.g. solvable group has polynomial or exponential growth. In 1972 Tits [32] obtained the same result for linear groups; this is famous as the so-called "Tits Alternative." In 1975 Adian [1] established that the growth of the free noncyclic Burnside group $B(m, n)$ of exponent n is exponential if $n \geq 665$ is an odd integer.

A powerful result was obtained by Gromov in 1981 [7]. He proved that every f.g. group of polynomial growth contains a nilpotent subgroup of finite index. In combination with an earlier result of Malcev [15] this implies that every f.g. group of polynomial growth satisfies some nontrivial semigroup identity. In 1987 Grigorchuk [11] obtained the same result for f.g. cancellative semigroups. The current author [28] gave the examples of f.g. semigroups having polynomial growth and not satisfying any nontrivial identity.

We note here another powerful result of Grigorchuk, who while solving a famous problem of Milnor's, gave in 1980−1983 [9, 10] the first example of a f.g. group having intermediate growth. Earlier Govorov [8] (his proof needs a slight correction) and Shearer [25] gave the first examples of f.g. and respectively finitely presented semigroups and associative algebras of intermediate growth. In 1993 Okninski [21] gave the first example of a f.g. linear semigroup with intermediate growth.

The possible types of growth for f.g. semigroups have been studied by Bergman [4] who proved, in particular, that if the growth of a f.g. semigroup is less than quadratic then it is linear. Trofimov [33] showed how to construct a continuum of two generated semigroups having quadratic growth and got some other results about the asymptotic behavior for the growth functions of f.g. semigroups. Some other interesting results about the semigroups of polynomial growth were obtained by Krempa and Okninski [14].

Here the focus of our attention will be semigroup identities and growth functions. Our goal is to study the asymptotic behavior of a semigroup satisfying a given system of identities.

Let **Ł** be a semigroup variety. It is natural to ask:

QUESTION 1. *What kind of growth can f.g. semigroups of the variety* **Ł** *have?*

First of all, it is natural to investigate the free objects. Thus, we have

QUESTION 2. *What kind of growth can f.g.* **Ł**-*free semigroups have*?

In particular, we can formulate

QUESTION 3. *Does there exist a relatively free semigroup having intermediate growth*?

Questions 1–3 have a direct relationship to the following.

Problem 1 (Sapir [24]). Give the description of all semigroup varieties in which every f.g. relatively free semigroup has polynomial growth.

We note that few results about the growth of relatively free semigroups are known. Besides Adian's paper [1] we want to mention the results of Frandendurg [5] and Brinkhuis [6] who independently proved that in the variety defined by the identity $X^2 = 0$ the 3-generated relatively free semigroup has exponential growth. A very nontrivial example of a relatively free semigroup of polynomial growth in the variety defined by the identity whose right side is a zero was given by Baker *et al.* [2].

One of the main purposes of this paper is to give examples of relatively free semigroups and monoids having intermediate growth (Section 6). The basic examples belong to the variety **B** which is nonperiodic and defined by the identity

$$xyuyx = yxuxy.$$

This identity was introduced in the paper of Neumann and Taylor [20] as one of the semigroup identities that every nilpotent group of class 2 satisfies. At the same time we answer (Section 7) Grigorchuk's question about the existence of a relatively free semigroup whose growth is not a polynomial but is strongly less than the growth of the function $p(m)$ of all partitions for a natural number m. We prove that the **B**-free semigroup of rank 3 has this property, and moreover its Gelfand–Kirillov superdimension equals zero. In Section 8 we show how to construct a continuum of nilpotent (in the sense of Neumann–Taylor) semigroups having intermediate growth and zero Gelfand–Kirillov superdimension. We also give examples of f.g. nil-semigroups belonging to a periodic variety (with the identity $x^2 = 0$) and having arbitrarily small intermediate growth.

In [17] Milnor gave a simple criterion for f.g. groups and semigroups to have exponential growth. Here, one of the main tools for proofs is some new points of view on growth (Section 3) that make it possible to formulate new growth criteria both for f.g. semigroups and for other algebraic systems. We also present a general method for constructing different types of examples of f.g. semigroups with intermediate growth (Section 4). In Section 5 we show how to find "subsets of intermediate growth" in a nonmonogenic free semigroup using the properties of the classical Fibonacci sequence. Combining these results and methods with some properties of the Neumann–Taylor [20] sequence of the semigroup identities, we get our main results (Theorems 6.1, 6.2, 7.1, and 8.1).

The main ideas underlying our proof together with the results obtained by Sapir in [23] allowed us to make further progress in our investigations. We obtained necessary and sufficient conditions which must be satisfied by a nonperiodic variety of semigroups over a finite set of variables in order for every f.g. semigroup of this variety to have polynomial growth. These results will be published in two forthcoming papers, "Growth, Unavoidable Words, and M. Sapir's Conjecture for Semigroup Varieties" and "Relatively Free Semigroups of Polynomial Growth." Quite recently our methods helped us to construct examples of relatively free semigroups of intermediate growth and maximal (equals 1) Gelfand–Kirillov superdimension.

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2. PRELIMINARIES

Let $\mathscr{C}$ be the set of all monotone nondecreasing functions from $\mathbf{N}$ into $\mathbf{R}^+$.

Define a binary relation $\preceq$ on the set $\mathscr{C}$. Let $f, g \in \mathscr{C}$. We shall write

$$f \preceq g$$

if there exist natural numbers k_1 and k_2 such that

$$f(m) \leq k_1 g(k_2 m) \qquad \textit{for}\, m = 1, 2, \dots .$$

For any functions $f, g \in \mathscr{C}$ put

$$f \rho g \Leftrightarrow f \preceq g \wedge g \preceq f.$$

Then ρ is an equivalence and for every function $[\mathbf{f}] \in \mathscr{C}$ the equivalence class $[f] \in \mathscr{C}/\rho$ is called the *growth* of the function f. The growth $[\mathbf{f}]$ is called *polynomial* if there exists a real positive number d such that $[\mathbf{f}] \preceq [\mathbf{m}^d]$.

The growth $[\mathbf{f}]$ is called *exponential* if $[\mathbf{f}] = [\mathbf{2}^m]$ and *subexponential* if $[\mathbf{f}] \prec [\mathbf{2}^m]$.

The growth $[\mathbf{f}]$ is called *intermediate* if $[\mathbf{m}^d] \prec [\mathbf{f}] \prec [\mathbf{2}^m]$ for an arbitrary real positive number d.

Let $S = \langle X \rangle$ be a semigroup with a finite set X of generators. Also, let $z \in S$ and $l(z)$ be a minimal number of factors in all representations of Z as a product of elements of S. We say that $l(Z)$ is a *length* of Z.

The function

$$g_S \colon \mathbf{N} \to \mathbf{N}$$

where

$$g_S(m) = \operatorname{card}\{z \in S : l(z) \leq m\}$$

is called the *growth function of the semigroup S*. The growth denoted $[\mathbf{g}_S]$ is called the *growth of the semigroup S*.

It follows from this definition that

(a) S has polynomial growth if and only if

$$g_S(m) \leq Cm^d \qquad (m = 1, 2, \dots)$$

for some real positive numbers C and d.

(b) S has exponential growth if and only if there exists a real positive number $c > 1$ such that

$$g_S(m) > c^m$$

for all sufficiently large numbers m.

Let H be a nonempty subset of S. We define the *growth function g_H of H* as a restriction of the function g_S on H.

$$g_H(m) = \operatorname{card}\{z \in H : l(z) \leq m\}$$

The growth $[\mathbf{g}_H]$ will be called the *growth of the subset H*. Such a definition makes it possible for us to speak about subsets of S having polynomial, exponential, or intermediate growth and also to consider the case where H is not finitely generated.

We shall use the sign $\equiv$ to designate the graphical equality of words over some alphabet.

Let r be a real number, then $[r]$ denotes the integral part of r.

As usual, by S^1 we shall denote the monoid of the semigroup S.

We mention the following condition, giving simple examples of the subsets of S with polynomial growth.

PROPOSITION 2.1. *Let S be a semigroup and $H \subseteq S$. Let us also assume that there exist some elements $a_1, a_2, \dots, a_k$, $c_1, c_2, \dots, c_k \in S^1$ such that every element $a \in H$ can be represented in the form*

$$a = c_1 a_1^{\alpha_1} c_2 a_2^{\alpha_2} \cdots c_k a^{\alpha_k} c_{k+1} \equiv V, \tag{1}$$

where $l(V) \le l(a)$. Then $[\mathbf{g}_H] \preceq [\mathbf{m}^k]$. If in the representation (1) *the word V is uniquely determined by a and if $\alpha_1, \alpha_2, \dots, \alpha_k$ can be arbitrary natural numbers then $[g_H] = [\mathbf{m}^k]$.*

We also need the following elementary proposition.

PROPOSITION 2.2. *Let S be a semigroup and $H_1, H_2, \dots, H_q$ subsets of S having polynomial growth of degrees $n_1, n_2, \dots, n_q$, respectively. Suppose also that*

$$S = H_1 \cup H_2 \cup \cdots \cup Hq.$$

Then

$$[g_S] = \max\left\{[\mathbf{g}_{\mathbf{H}_1}], [\mathbf{g}_{\mathbf{H}_2}], \dots, [\mathbf{g}_{\mathbf{H}_q}]\right\}.$$

As above, let S be a semigroup with a finite fixed set X of generators and F_X a free semigroup over the alphabet X. Clearly, we may represent the elements of S using the words from F_X. We also write F_k instead of F_X in the case when $|X| = k$.

Let us consider the lexicographic order $\preceq$ on F_X.

DEFINITION 2.1. The word $U \in S$ will be called *shortlex reduced* (relative to S) if for any word $V \in F_X$ such that $U = V$ in S we have that $l(U) \le l(V)$ and if $l(U) = l(V)$ then $U \preceq V$ in F_X.

3. GROWTH CRITERIA

In this section we give new points of view on growth connected with the notion of the attainment function. These make it possible to formulate new growth criteria for both semigroups and other algebraic systems.

Let $f(n)$ be a monotone increasing map from $\mathbf{N}$ into $\mathbf{N}$. Define a monotone nondecreasing map $T_f(m)$ from $\mathbf{N}$ onto $\mathbf{N}$ by the rule

$$T_f(m) = \min\{n : f(n) \ge m\}.$$

Let $f(i) = m_i$ $(i = 1, 2, \dots)$. Then

$$T_f(m) = \begin{cases} 1 & \text{for } 1 \le m \le m_1 \\ 2 & \text{for } m_1 < m \le m_2 \\ \cdot & \dots \\ k & \text{for } m_{k-1} < m \le m_k \end{cases},$$

and for any natural number m

$$T_f(f(m)) = m.$$

Define a function $h_f(x)$ of a real positive argument x by the same rule:

$$h_f(x) = \min\{n : f(n) \ge x\}.$$

We say that the function $h_f(x)$ is a step function corresponding to $f(m)$.

The following follows immediately from the definition of $h_f(x)$ and the elementary properties of the step functions.

PROPOSITION 3.1. *For any natural number n,*

$$\sum_{i=1}^{n} T_f(i) = 1 + \int_1^n h_f(x)\,dx. \tag{2}$$

As above, let S be a semigroup with a finite fixed set X of generators and F_X a free semigroup over the alphabet X. Again, let $\preceq$ be a lexicographic order on F_X and let

$$s_1 \prec s_2 \prec \cdots \prec s_k \prec \cdots \tag{3}$$

be the chain of all shortlex-reduced (relative to S) elements of F_X.

The following proposition follows easily from our previous definitions.

PROPOSITION 3.2. *For any natural number n,*

$$T_{g_S}(n) = l(s_n). \tag{4}$$

DEFINITION 3.1. Let r be a natural number. The number r is said to be *m-reached relative to S* (or for short, *m-reached*) if there exist different elements $U_1, U_2, \dots, U_r \in S$ such that

$$\sum_{j=1}^{r} l(U_j) \le m.$$

DEFINITION 3.2. Let $\Phi_S(m)$ be the set of all m-reached-relative-to-S numbers and $r_S(m)$ the maximal element of $\Phi_S(m)$. Clearly, we may consider $r_S(m)$ as a monotone nondecreasing function from $\mathbf{N}$ into $\mathbf{N}$. We will call this function the *attainment function* of the semigroup S.

The step function $h_{g_S}(x)$ corresponding to the growth function $g_S(m)$ will be denoted by $h_S(x)$ for simplicity.

Let $r \in \mathbf{N}$.

LEMMA 3.1. *The following conditions are equivalent*:

(i) *The number r is m-reached relative to S.*

(ii) *In the chain* (3), $\sum_{j=1}^{r} l(s_j) \leq m$.

(iii) $1 + \int_1^r h_S(x)\,dx \leq m$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from the obvious fact that for any elements $U_1, U_2, \ldots, U_r \in S$,

$$\sum_{j=1}^{r} l(s_j) \leq \sum_{j=1}^{r} l(U_j).$$

The equivalence (ii) $\Leftrightarrow$ (iii) is a simple consequence of the relations (2) and (4).

The lemma is proved.

Remark 3.1. It is easy to see that we may transfer the notion of an attainment function from a f.g. semigroup S to any of its subsets and the analogue of Lemma 3.1 will hold.

Again, let $f(m)$ be a monotone increasing map from $\mathbf{N}$ into $\mathbf{N}$. Define a new function

$$r_f \colon \mathbf{N} \to \mathbf{N}$$

by the rule

$$r = r_f(m) = \max\left\{k : \sum_{j=1}^{k} T_f(j) \leq m\right\}. \tag{5}$$

It follows from (2) and (5) that

$$r_f(m) = \max\left\{r : 1 + \int_1^r h_f(x)\,dx \leq m\right\}. \tag{6}$$

We also need the following three elementary lemmas.

LEMMA 3.2. *Let $f(m)$ be as above and $\varphi(x)$ a positive monotone increasing function of a real argument x which is continuous on the interval* $[1, +\infty)$ *and satisfies the conditions*

$$h_f(x) \leq \varphi(x) \leq 2h_f(x). \tag{7}$$

Let

$$L(m) = \max\left\{r : \int_1^r \varphi(x)\, dx \leq m\right\}. \tag{8}$$

Then the function $L(m)$ has the same growth as $r_f(m)$.

Proof. It follows from (7) that for any natural number $t \in N$,

$$\int_1^t h_f(x)\, dx \leq \int_1^t \varphi(x)\, dx \leq \int_1^t 2h_f(x)\, dx. \tag{9}$$

Putting in (9) $t = r = r_f(m)$, we obtain that

$$\int_1^{r_f(m)} h_f(x)\, dx \leq \int_1^{r_f(m)} \varphi(x)\, dx \leq \int_1^{r_f(m)} 2h_f(x)\, dx. \tag{10}$$

Let us fix a number m and, as above, let $r = r_f(m)$. First note that

$$\sum_{i=1}^{r} T_f(i) > \frac{m}{2}. \tag{11}$$

Indeed, suppose to the contrary and let $\sum_{i=1}^{r} T_f(i) \leq m/2$. Then $T_f(r) < m/2$ and since $T_f(r+1) \leq T_f(r) + 1$ we have that $T_f(r+1) < m/2 + 1$. So $\sum_{i=1}^{r+1} T_f(i) \leq m$. This contradicts (see Formula (5)) the definition of the number r.

Put

$$\theta_1(m) = \min\left\{p \in \mathbf{N} : \sum_{i=1}^{p} T_f(i) > \frac{m}{2}\right\}.$$

The inequality (11) shows that

$$\theta_1(m) \leq r_f(m) \qquad (m = 1, 2, \dots). \tag{12}$$

Clearly, at the same time

$$r_f(m) < \theta_1(2m). \tag{13}$$

It follows immediately from (12) and (13) that

$$[\mathbf{r}_f(\mathbf{m})] = [\theta_1(\mathbf{m})]. \tag{14}$$

Now, let $t \in \mathbf{N}$ and $\int_1^t \varphi(x)\,dx \geq m$. Combining (2) and (9) we obtain $\int_1^t \varphi(x)\,dx < 2\sum_{i=1}^t T_f(i)$. So, $\sum_{i=1}^t T_f(i) > m/2$.

Put now $\tilde{t} = L(m)$, where $L(m)$ is from (8). Then $\int_1^{\tilde{t}-1} \varphi(x)\,dx < m$, and by taking into account (2) and (9) we obtain $\sum_{i=1}^{\tilde{t}-1} T_f(i) < m$. Thus, $\tilde{t} \leq r_f(m)$ and in (8)

$$L(m) \leq r_f(m). \tag{15}$$

On the other side, by (10) we have that

$$\theta_1(m) \leq L(m). \tag{16}$$

Combining (14), (15), and (16) we obtain

$$[\mathbf{L}(\mathbf{m})] = [\mathbf{r}_f(\mathbf{m})].$$

The lemma is proved.

LEMMA 3.3. *Let $f(m), g(m)$ be monotone increasing functions from $\mathbf{N}$ into $\mathbf{N}$. Then*

$$f(m) \preceq g(m) \Leftrightarrow T_g(m) \preceq T_f(m).$$

In particular,

$$[\mathbf{f}] = [\mathbf{g}] \Leftrightarrow [\mathbf{T}_f] = [\mathbf{T}_g].$$

Proof. First show that

$$f(m) \preceq g(m) \Rightarrow T_g(m) \preceq T_f(m).$$

Suppose to the contrary. Then there exist $C_1, C_2 > 0$ such that

$$f(m) \leq C_1 g(C_2 m) \qquad (m = 1, 2, \dots), \tag{17}$$

and at the same time for any $D_1, D_2 > 0$ there exists a natural number $\overline{m}$ such that

$$T_g(\overline{m}) \geq D_1 T_f(D_2 \overline{m}). \tag{18}$$

Also, let q be a natural number such that

$$T_f(D_2 \overline{m}) = q + 1.$$

Then (18) can be written in the form

$$T_g(\overline{m}) \geq D_1(q + 1). \tag{19}$$

In view of (17) we also have

$$f(q+1) \leq C_1 g\big(C_2(q+1)\big).$$

Since $f(T_f(D_2\overline{m}) \geq D_2\overline{m}$ we obtain

$$D_2\overline{m} \leq C_1 g\big(C_2(q+1)\big). \tag{20}$$

Put now

$$D_2 > C_1.$$

Then, by virtue of (20), we obtain

$$\overline{m} \leq g\big(C_2(q+1)\big).$$

So,

$$T_g(\overline{m}) \leq T_g\big(g\big(C_2(q+1)\big) = C_2(q+1),$$

and if we put $D_1 > C_2$ we will have a contradiction with the inequality (19).

Now let us prove the implication

$$T_g(m) \preceq T_f(m) \Rightarrow f(m) \preceq g(m). \tag{21}$$

Suppose to the contrary. Then for some $k_1, k_2 \in \mathbf{N}$ we have that

$$T_g(m) \leq k_1 T_f(k_2 m) \qquad (k_1, k_2 > 0)\ m = 1, 2, \dots, \tag{22}$$

and at the same time for any $C_1, C_2 > 0$ there exists a natural number $\overline{m}$ such that

$$f(\overline{m}) > C_1 g(C_2\overline{m}). \tag{23}$$

Put now

$$C_1 > k_2, \qquad C_2 > k_1.$$

Then

$$T_g\big(g(C_2\overline{m}\big) = C_2\overline{m} > k_1\overline{m}. \tag{24}$$

On the other hand, in view of (22) and (23) we obtain

$$\begin{aligned} T_g\big(g\big(C_2(\overline{m})\big) &\leq k_1 T_f\big(k_2 g\big(C_2\overline{m}\big)\big) \\ &\leq k_1 T_f\big(C_1 g\big(C_2(\overline{m})\big) \leq k_1 T_f\big(f(\overline{m})\big) = k_1\overline{m}. \end{aligned}$$

This contradicts the inequality (24). So, the implication (21) holds.

The lemma is proved.

LEMMA 3.4. *As above, let $f(m)$, $g(m)$ be monotone increasing maps from* $\mathbf{N}$ *into* $\mathbf{N}$. *Then*

$$T_g(m) \preceq T_f(m) \Leftrightarrow r_f(m) \preceq r_g(m). \tag{25}$$

In particular,

$$[\mathbf{T}_f] = [\mathbf{T}_g] \Leftrightarrow [\mathbf{r}_f] = [\mathbf{r}_g].$$

Proof. First, show that the implication

$$T_g \preceq T_f \Rightarrow r_f \preceq r_g \tag{26}$$

holds.

Suppose to the contrary. Then for some $k_1, k_2 \in \mathbf{N}$,

$$T_g(m) \leq k_1 T_f(k_2 m) \qquad (m = 1, 2, \dots), \tag{27}$$

and at the same time for any $C_1, C_2 > 0$ there exists a natural number $\overline{m}$ such that

$$r_f(\overline{m}) > C_1 r_g(C_2 \overline{m}). \tag{28}$$

Let $\bar{r} = r_g(C_2 \overline{m})$. Then

$$\sum_{i=1}^{\bar{r}+1} T_g(i) > C_2 \overline{m} \tag{29}$$

and simultaneously, in view of the inequality (28),

$$\sum_{i=1}^{C_1 \bar{r}} T_f(i) \leq \overline{m}. \tag{30}$$

Put $C_1 > k_2$. Then, taking into account (30),

$$\sum_{i=1}^{\bar{r}} T_f(ik_2) < \sum_{i=1}^{C_1 \bar{r}} T_f(i) \leq m.$$

Therefore,

$$k_1 \sum_{i=1}^{\bar{r}} T_f(ik_2) < k_1 \overline{m},$$

and by virtue of (27)

$$\sum_{i=1}^{\bar{r}} T_g(i) < k_1 \overline{m}. \tag{31}$$

Now put $C_2 > 2k_1$. Since $T_g(r+1) \le 1 + T_g(r)$, using (29) we obtain

$$\sum_{i=1}^{\bar{r}} T_g(i) \ge \frac{C_2}{2}\overline{m} > k_1\overline{m}.$$

This contradicts (31) and completes the proof of the implication (26).

Now we give the proof of the implication

$$r_f \preceq r_g \Rightarrow T_g(m) \preceq T_f(m). \tag{32}$$

Suppose that there exist $k_1, k_2 \in \mathbf{N}$ such that

$$r_f(m) \le k_1 r_g(k_2 m) \qquad (m = 1, 2, \dots) \tag{33}$$

and at the same time for any $C_1, C_2 > 0$ there exists an $\overline{m} \in \mathbf{N}$ such that

$$T_g(\overline{m}) \ge C_1 T_f(C_2\overline{m}). \tag{34}$$

The inequality (33) means that if $r, m \in \mathbf{N}$ and

$$\sum_{i=1}^{r} T_g(i) \le k_2 m$$

but

$$\sum_{i=1}^{r+1} T_g(i) > k_2 m,$$

then

$$\sum_{i=1}^{k_1 r} T_f(i) > m.$$

Now choose $C_1 > 5k_1k_2$ and let d_1 be a natural number such that

$$4k_1k_2 < d_1 < C_1. \tag{35}$$

Put $C_2 > d_2 q$, where d_2, $q \in \mathbf{N}$, $q > 1$, and

$$d_2 > 2k_1. \tag{36}$$

Then, in view of (34) we have that

$$T_g(\overline{m}) > d_1 T_f(d_2\overline{m})$$

$$T_g(\overline{m}+1) > d_1 T_f(d_2(\overline{m}+1)$$

$$T_g(\overline{m}+2) > d_1 T_f(d_2\overline{m}+2)$$

$$\dots$$

$$T_g(q\overline{m}) > d_1 T_f(qd_2\overline{m}).$$

So,

$$\sum_{i=\overline{m}}^{q\overline{m}} T_g(i) > d_1 \sum_{i=\overline{m}}^{q\overline{m}} T_f(d_2 i). \tag{37}$$

Since $T_f(m)$ is a monotone nondecreasing function we have that

$$\sum_{i=1}^{\overline{m}} \mathbf{T}_f(d_2 i) \leq \frac{1}{q} \sum_{i=1}^{q\overline{m}} T_f(d_2 i).$$

Therefore

$$\sum_{i=\overline{m}}^{q\overline{m}} T_f(d_2 i) \geq \frac{q-1}{q} \sum_{i=1}^{q\overline{m}} T_f(d_2 i). \tag{38}$$

Put now

$$m = 2k_1 \left(\sum_{i=1}^{q\overline{m}} T_f(d_2 i) \right)$$

and $\hat{r} = q\overline{m}$.

Then, taking into account (35), (37), and (38) we have that

$$\sum_{i=1}^{\hat{r}} T_g(i) > d_1 \frac{q-1}{q} \frac{m}{2k_1} \geq d_1 \frac{m}{4k_1} > k_2 m. \tag{39}$$

At the same time,

$$\sum_{i=1}^{k_1\hat{r}} T_f(i) = \sum_{i=1}^{k_1 q\overline{m}} T_f(i) = \sum_{i=1}^{q\overline{m}} \sum_{j=1}^{k_1} T_f\bigl(j + (k_1(i-1))\bigr). \tag{40}$$

Using a nondecreasing property of $T_f(m)$ and the inequality (36) we obtain

$$\sum_{j=1}^{k_1} T_f\bigl(j + (k_1(i-1))\bigr) \leq k_1 T_f(k_1 i) \leq k_1 T_f(d_2 i). \tag{41}$$

So, by (40) and (41),

$$\sum_{i=1}^{k_1\hat{r}} T_f(i) \leq k_1 \sum_{i=1}^{q\overline{m}} T_f(d_2 i) = \frac{m}{2} < m. \tag{42}$$

It follows from (39) that $r_g(k_2 m) < \hat{r}$. By (42) we have $r_f(m) > k_1\hat{r}$. Therefore,

$$r_f(m) > k_1 r_g(k_2 m).$$

This contradicts (33) and completes the proof of the implication (32). The lemma is proved.

By Lemmas 3.3 and 3.4 we obtain

COROLLARY 3.1. *Let $f(m)$, $g(m)$ be monotone increasing functions from* $\mathbf{N}$ *into* $\mathbf{N}$. *Then* $[\mathbf{f}] = [\mathbf{g}] \Leftrightarrow [\mathbf{r}_f] = [\mathbf{r}_g]$.

Now we consider some other important corollaries following from Lemmas 3.1–3.4. We start with the following statement which is a simple combination of these lemmas.

THEOREM 3.1. *Let S and K be f.g. semigroups. Then S and K have the same growth if and only if their attainment functions $r_S(m)$ and $r_K(m)$ have the same growth.*

THEOREM 3.2. *Let S be a f.g. semigroup. S has exponential growth if and only if the growth of attainment function of S equals* $[\mathbf{m}/\ln \mathbf{m}]$. *In other words,*

$$[g_S] = [2^{\mathbf{m}}] \Leftrightarrow [r_S] = \left[\frac{\mathbf{m}}{\ln \mathbf{m}}\right].$$

Proof. By Theorem 3.1, it suffices to consider the case when $S = F_k$ is the free semigroup of a finite rank $k > 1$. Let $g(m) = g_{F_k}(m)$ be a growth function of F_k. Since

$$g_{F_k}(m) = \sum_{i=1}^{m} k^i = \frac{k^{m+1} - k}{k - 1}$$

we have that

$$T_g(m) - 1 < \log_k m < T_g(m) + 1.$$

Then the corresponding step function $h_{F_k}(x)$ of the semigroup F_k satisfies the inequalities

$$h_{F_k}(x) - 1 < \log_k x < 1 + h_{F_k}(x) \tag{43}$$

and by Lemma 3.2 we obtain that

$$\left[\mathbf{r}_{F_k}(\mathbf{m})\right] = [\mathbf{L}(\mathbf{m})],$$

where

$$L(m) = \max\left\{r : \int_1^r \log_k x\, dx \le m\right\}.$$

Clearly, for any fixed number θ such that $0 < \theta < 1$ and for all sufficiently large numbers m,

$$\frac{m \ln k}{\ln m}(1-\theta) < L(m) < \frac{m \ln k}{\ln m}(1+\theta).$$

Thus

$$L(m) \sim \frac{m}{\ln m} \ln k = \frac{m}{\log_k m} \quad (\text{as } m \to \infty)$$

and we obtain that

$$\left[\mathbf{r}_{\mathbf{F}_k}(m)\right] = \left[\frac{\mathbf{m}}{\ln \mathbf{m}}\right].$$

The theorem is proved.

Note that by combining (6) and (43) we obtain

$$r_{F_k}(m) \sim L(m) \ (\text{as } m \to \infty),$$

giving the formula for the asymptotic behavior of the attainment function of the semigroup F_k. More exactly, we have

COROLLARY 3.2. *As above, let F_k be a free semigroup of a finite rank $k > 1$. Then*

$$r_{F_k}(m) \sim \frac{m}{\log_k m} \quad (\text{as } m \to \infty).$$

Using Corollary 3.1 and Theorem 3.2 we obtain

COROLLARY 3.3. *The growth of a f.g. semigroup S is subexponential if and only if*

$$[\mathbf{r}_{\mathbf{S}}(m)] \prec \left[\frac{\mathbf{m}}{\ln \mathbf{m}}\right].$$

THEOREM 3.3. *Let S be an f.g. infinite semigroup. Then $[\mathbf{g}_S] = [\mathbf{m}^d]$ if and only if $[\mathbf{r}_{\mathbf{S}}] = [\mathbf{m}^{d/(d+1)}]$.*

Proof. Suppose that $[\mathbf{g}_S] = [\mathbf{m}^d]$. Note that since S is an infinite, $d \geq 1$ and so $f(m) = m^d$ is a monotone increasing function. Since $[\mathbf{g}_S] = [\mathbf{f}]$, by Corollary 3.1 it suffices to check the equality

$$[\mathbf{r}_f] = [\mathbf{m}^{d/(d+1)}].$$

Let $m \in \mathbf{N}$ and also let

$$k^d < m \leq (k+1)^d$$

for some natural number k. Then, $T_f(m) = k + 1$ and

$$(m)^{1/d} \leq T_f(m) < (m)^{1/d} + 1.$$

Using Lemma 3.2 we obtain

$$[\mathbf{r}_f] = [\mathbf{L}_d] \tag{44}$$

where

$$L_d(m) = \max\left\{ r : \int_1^r x^{1/d}\, dx < m \right\}.$$

Clearly, if m is sufficiently large that

$$m^{d/(d+1)} < L_d(m) < 2m^{d/(d+1)}.$$

This means that

$$[\mathbf{L_d}] = [\mathbf{m}^{d/(d+1)}],$$

and taking into account (44) and Theorem 3.1 we get the conclusion of our theorem.

The theorem is proved.

By Theorems 3.1–3.3 we obviously have

THEOREM 3.4. *Let S be an f.g. infinite semigroup. S has polynomial growth if and only if there exists a real number ζ such that $\frac{1}{2} \leq \zeta < 1$ and*

$$[\mathbf{r}_S] \prec [\mathbf{m}^{\zeta}].$$

The following result follows immediately from Theorems 3.2 and 3.4.

THEOREM 3.5. *A finitely generated semigroup S has intermediate growth if and only if for any real number ζ such that $0 < \zeta < 1$,*

$$[\mathbf{m}^{\zeta}] \prec [\mathbf{r}_S] \prec \left[\frac{\mathbf{m}}{\ln \mathbf{m}}\right].$$

Remark 3.2. Since the proofs of Theorems 3.1–3.5 are based on Lemmas 3.1–3.4, taking into account Remark 3.1 the statements of these theorems and their corollaries become true if instead of f.g. semigroups we speak on their arbitrary subsets.

For instance, as above, let S be a f.g. semigroup with a fixed set of generators and let $H \subseteq S$.

Then H has exponential growth $\Leftrightarrow [\mathbf{r_H}] = [\mathbf{m}/(\ln \mathbf{m})]$, H has subexponential growth $\Leftrightarrow [\mathbf{r_H}] \prec [\mathbf{m}/(\ln \mathbf{m})]$, etc.

4. CONSTRUCTIONS

Let m be a natural number. Let us denote by $p(m)$, $p_r(m)$, and $q_r(m)$ the numbers of all partitions, all r-partitions, and all ordered r-partitions respectively for a number m. It is well known that

$$q_r(m) \leq r!p(m) \tag{45}$$

and by the classical result of Hardy–Ramanujan that $[\mathbf{p(m)}] = [\mathbf{2}^{\sqrt{m}}]$.

We also need the following elementary proposition.

PROPOSITION 4.1. *Let $r, m \in \mathbf{N}$ and for some real numbers $\gamma > 0$ and $c > 1$,*

$$r < \frac{m\gamma}{\log_c m}.$$

If m is sufficiently large then

$$r! < c^{(\gamma+1)m}.$$

As above, let S be a semigroup with a finite generating set X. Also, let $H \subseteq S$ and $H_m \subseteq S$ be the set of all words from H having length m relative to X.

Put

$$E_H^{(r,m)} = \left\{ (u_1, u_2, \dots, u_r) : u_j \in H,\ \sum_{j=1}^{r} l(u_j) = m \right\}, \tag{46}$$

$$\overline{E}_H^{(r,m)} = \left\{ (u_1, u_2, \dots, u_r) : u_j \in H,\ \sum_{j=1}^{r} l(u_j) \leq m \right\}. \tag{47}$$

LEMMA 4.1. *Let β, c be real positive numbers such that*

$$\operatorname{card} H_m < \beta c^m \qquad (m = 1, 2, \dots). \tag{48}$$

Then

$$\operatorname{card} E_H^{(r,m)} < \beta^r c^m q_r(m), \tag{49}$$

$$\operatorname{card} \overline{E}^{(r,m)} < m \beta^r c^m q_r(m). \tag{50}$$

Proof. Let

$$u = (u_1, u_2, \dots, u_r) \in E_H^{(r,m)},$$
$$l_j = l(u_j) \qquad (j = 1, 2, \dots).$$

Then by (46)

$$\sum_{j=1}^{r} l_j = m. \tag{51}$$

In view of (48) the elements u_j satisfying (46) could be chosen in less than

$$\beta c^{l_1} \beta c^{l_e} \cdot \dots \cdot \beta c^{l_r} = \beta^r c^m$$

different ways.

Since the total number of all possible ordered finite sequences

$$(l_1, l_2, \dots, l_r)$$

satisfying condition (51) is exactly equal to $q_r(m)$ we obtain the inequality (49).

Let

$$O(m) = \operatorname{card} E_H^{(r,m)}, \qquad \overline{O}(m) = \operatorname{card} \overline{E}_H^{(r,m)}$$

be mappings from **N** into **N**.

Since $O(m)$ is an increasing function and

$$\overline{O}(m) = \sum_{j=1}^{m} O(j),$$

we obtain that

$$\overline{O}(m) < mO(m). \tag{52}$$

Inequality (50) now follows immediately from (49) and (52).

The lemma is proved.

Let S be as above and $H, K \subseteq S$. Also, let $\Upsilon_r(H, K)$ be the set of all shortlex-reduced (relative to the semigroup S) words of the type

$$W \equiv h_1 v_1 h_2 v_2 \cdots h_r v_r h_{r+1}, \tag{53}$$

where $h_1, h_{r+1} \in H^1 = \{H \cup 1\}$, $h_2, \dots, h_r \in H$, $v_1, v_2, \dots, v_r \in K$. In this case, we say that

$$v_1, v_2, \dots, v_r \tag{54}$$

are the *main factors* of W.

Let

$$\Upsilon(H, K) = \bigcup_{r=1}^{\infty} \Upsilon_r(H, K).$$

It is easy to see that

$$\Upsilon(H, K) = \Upsilon(K, H).$$

Let L be a natural number.

DEFINITION 4.1. The set $\Upsilon(H, K)$ is called *L-bounded* if, for any word W (53) from this set, a finite sequence of its main factors (54) does not contain more than L identical elements.

In particular, the set $\Upsilon(H, K)$ is 1-bounded if for any $W \in \Upsilon(H, K)$ in the representation (53) all factors (54) are different.

DEFINITION 4.2. Let $\mathscr{A}_L(H, K)$ be a union of all L-bounded subsets of the set $\Upsilon(H, K)$. In this case we say that $\mathscr{A}_L(H, K)$ is the L-bounded product of subsets H and K.

Clearly, $\mathscr{A}_L(H, K)$ is a maximal L-bounded subset of $\Upsilon(H, K)$.

LEMMA 4.2. *Let $H, K \subseteq F_X$ and the growths of both H and K be subexponential. Then for any natural number L the growth of the set $\mathscr{A}_L(H, K)$ is also less than exponential.*

Proof. Since both H and K have subexponential growth, we have that for any number $c > 1$ and for all sufficiently large numbers m the inequalities

$$\operatorname{card}(H)_m, \operatorname{card}(K)_m < c^m$$

hold. In particular, there is a number β such that

$$\operatorname{card}(H)_m, \operatorname{card}(K)_m < \beta c^m \qquad (m = 1, 2, \dots). \tag{55}$$

Let $m \in \mathbf{N}$, $W \in \mathscr{A}_L(H, K)$, and again let (53) be a representation of W with L or fewer identical main factors (54). Now put

$$h = (h_1, h_2, \dots, h_{r+1}),$$
$$v = (v_1, v_2, \dots, v_r).$$

Let $l(W) = m$, then

$$h \in \overline{E}_H^{(r+1, m)}, \qquad v \in \overline{E}_K^{(r, m)}. \tag{56}$$

Let $D_r \subseteq \mathscr{A}_L(H, K)$ be an L-bounded subset consisting of all words given by the presentation (53) with a fixed number r.

It is obvious that

$$\operatorname{card}\big(\mathscr{A}_L(H, K)\big)_m = \sum_{r=1}^{m} \operatorname{card}(D_r)_m. \tag{57}$$

Since the word W is uniquely determined by h and v, in view of (56) we have that

$$\begin{aligned}\operatorname{card}(D_r)_m &= \operatorname{card}\{W : W \in D_r \text{ and } l(W) = m\} \\ &< \operatorname{card}\big(\overline{E}_H^{(r+1,m)}\big)\operatorname{card}\big(\overline{E}_K^{(r,m)}\big).\end{aligned}$$

Combining (55) and Lemma 4.1 we obtain

$$\operatorname{card}(D_r)_m < \beta^{r+1} m c^m q_{r+1}(m) \cdot \big(m\beta^r c^m q_r(m)\big).$$

Therefore,

$$\operatorname{card}(D_r)_m < m^2 \beta^{3r} c^{2m} q_{r+1}^2(m),$$

and by virtue of (45)

$$\operatorname{card}(D_r)_m < [(r+1)!]^2 (p(m))^2 m^2 \beta^{3r} c^{2m}. \tag{58}$$

Let $r_K(m)$ be the attainment function of the set K and $W \in \mathscr{A}_L(H, K)$ be a word with a graphical representation (53). As above, let $l(W) = m$. It follows from the definition of the subset $\mathscr{A}_L(H, K)$ that for any $W \in \mathscr{A}_L(H, K)$ in the representation (53) there are more than $r/2L$ different main factors (54). This shows that

$$\frac{r+1}{2L} \le r_K(m). \tag{59}$$

Since the growth of the set K is less than exponential, by Remark 3.2 we have that for any $c > 1$ and for all sufficiently large numbers m,

$$r_K(m) < \frac{2m}{\log_c m}.$$

Now the inequality (59) implies that

$$r + 1 < \frac{4Lm}{\log_c m}. \tag{60}$$

At the same time, in view of (57) and (58)

$$\operatorname{card}(\mathscr{A}_L(H,K))_m < \sum_r [(r+1)!]^2 (p(m))^2 m^2 \beta^{3r} c^{2m}. \tag{61}$$

Combining (60), (61), and Proposition 4.1 we obtain

$$\operatorname{card}(\mathscr{A}_L(H,K))_m < \sum_r c^{(4L+1)m} (p(m))^2 m^2 \beta^{12(Lm/(\log_c m))} c^{2m},$$

and taking into account the Hardy–Ramanujan formula,

$$\operatorname{card}(\mathscr{A}_L(H,K))_m < c^{8Lm}$$

for all sufficiently large numbers m.

Since c is an arbitrary number which is greater than 1, the last inequality shows that the growth of the set $\mathscr{A}_L(H,K)$ is strongly less than exponential.

The lemma is proved.

Let S be a f.g. semigroup and L be a natural number. We say that the set $G \subseteq S$ is *L-bounded* if S contains two subsets H and K of subexponential growth such that $G \subseteq \mathscr{A}_L(H,K)$. We say that G is *bounded* if G is L-bounded for some number L.

The following follows obviously from Lemma 4.2.

COROLLARY 4.1. *Let S be a f.g. semigroup. Then every bounded subset of S has subexponential growth.*

Let S be a finitely generated semigroup with a finite system of generators.

$$a_1, a_2, \ldots, a_n. \tag{62}$$

Also, let $W \in S$ and $k = \max\{i : a_i \text{ occurs in } W\}$. Then W can be written in the form

$$W \equiv a_k^{\alpha_0} W_1 a_k^{\alpha_1} W_2 \cdots a_k^{\alpha_{q-1}} W_q a_k^{\alpha_q}, \tag{63}$$

where

$$W_1, W_2, \ldots, W_q \tag{64}$$

are some words over the alphabet $\{a_1, a_2, \ldots, a_{k-1}\}$. Let S_1 and S_2 be the subsemigroups of S generated by a_k and $a_1, a_2, \ldots, a_{k-1}$, respectively. Clearly, if the word W is shortlex-reduced relative to S then $W \in \Upsilon(S_1, S_2)$ and we may consider the subwords (64) as the main factors of W (with respect to S_1 and S_2).

As above, let L be a fixed natural number.

Definition 4.3. We say that a semigroup S satisfies the condition M_L if every element $W \in S$ can be written in the shortlex-reduced form

$$W = a_k^{\beta_0}\overline{W}_1 a_k^{\beta_1}\overline{W}_2 \cdots a_k^{\beta_{p-1}}\overline{W}_p a_k^{\beta_p} \equiv \overline{W}, \tag{65}$$

where the number of identical main factors for $\overline{W}$ is never greater than L.

Theorem 4.1. *Let S be a f.g. semigroup satisfying the condition M_L for some natural number L. Then S has subexponential growth.*

Proof. Indeed, let (62) be a set of generators for S. Also, let S_1 and S_2 be the subsemigroups of S generated by a_n and $a_1, a_2, \dots, a_{n-1}$, respectively. Since S satisfies the M_L condition, in view of (65) $S = \mathscr{A}_L(S_1, S_2)$. Since S_1 is a monogenic subsemigroup it has linear growth. Then by Lemma 4.2 we obtain that S has subexponential growth if and only if S_2 has subexponential growth. Taking into account that S_2 is an $(n-1)$-generated semigroup, we can finish our proof using a simple induction by n.

The theorem is proved.

5. ON SUBEXPONENTIAL SUBSETS OF A FREE SEMIGROUP F_X

As above, let F_X be a free semigroup over finite set X. In this section we introduce some subsets of F_X having intermediate growth. The results we obtain will help us find sufficient conditions for semigroup varieties to contain semigroups whose growth is greater than polynomial. These conditions are closely connected with some elementary properties of the classical Fibonacci number sequence $\{\mathsf{F}_n\}$,

$$1, 1, 2, 3, 5, 8, 13, \dots,$$

defined by the rule $\mathsf{F}_1 = \mathsf{F}_2 = 1$, $\mathsf{F}_{n+2} = \mathsf{F}_{n+1} + \mathsf{F}_n$, and the number

$$\varphi = \lim_{n\to\infty} \frac{\mathsf{F}_{n+1}}{\mathsf{F}_n} = \frac{1+\sqrt{5}}{2}.$$

Let

$$X = \{x_1, x_2, \dots, x_p\} \qquad (p > 2) \tag{66}$$

and also let $W \in F_X$ be a word of the type

$$W \equiv \prod_j^k \left(x_1^{\alpha_j} x_2^{\alpha_j} \cdots x_q^{\alpha_j}\right) \qquad (2 < q \le p). \tag{67}$$

DEFINITION 5.1. The word W is called *positively directed* if on the right-hand side of the graphical equality (67)

$$\alpha_1 < \alpha_2 < \cdots < \alpha_k, \tag{68}$$

and W is called *negatively directed* if in (67)

$$\alpha_1 > \alpha_2 > \cdots > \alpha_k.$$

DEFINITION 5.2. A positively (negatively) directed word W (67) is said to be a *word of Fibonacci type* if in (67)

$$\alpha_{j+2} > \alpha_j + \alpha_{j+1}, \tag{69}$$

respectively,

$$\alpha_j > \alpha_{j+1} + \alpha_{j+2}. \tag{70}$$

Remark 5.1. It is easy to see that W is a positively directed word of Fibonacci type if and only if the word

$$W^* \equiv \prod_{j=1}^{k} \left(x_q^{\alpha_{k-j+1}} x_{q-1}^{\alpha_{k-j+1}} \cdots x_1^{\alpha_{k-j+1}} \right) \tag{71}$$

is a negatively directed word of Fibonacci type. We also note that $l(W) = l(W^*)$.

LEMMA 5.1. *Let W be a word of Fibonacci type. Also, let $m = l(W)$. Then on the right-hand side of the graphical equality* (67)

$$k < 5 + \log_{(1+\sqrt{5})/2} m.$$

Proof. (a) Suppose that the word W is positively directed.

It follows obviously from (69) that on the right-hand side of the equality (67)

$$\alpha_{q+5} > F_{q+1}\alpha_3 + F_{q+2}\alpha_4.$$

Let $\varphi = (1 + \sqrt{5})/2$. It is well known that

$$F_n < \varphi^n \qquad (n = 1, 2, \ldots),$$

and at the same time

$$F_n > \varphi^{n-3} \qquad (n = 3, 4, \ldots).$$

Thus

$$\alpha_k > F_{k-4}\alpha_3 + F_{k-3}\alpha_4 > F_{k-2}\alpha_3 > \varphi^{k-5}\alpha_2 \qquad (k > 5).$$

Since $l(W) = m$, we have that $\alpha_k < m/q$. Then on the right-hand side of (67),

$$k - 5 < \log_\varphi\left(\frac{m}{q}\right) < \log_\varphi m.$$

(b) Now, let W be a negatively directed word of Fibonacci type. Then by Remark 5.1 W^* is a positively directed word of Fibonacci type. So this case is reduced to the case (a) which has just been considered.

The lemma is proved.

Let $Ç \subset F_X$ be the set of all words of Fibonacci type over a finite alphabet X. Also, let $Ç_m \subset Ç$ be the of all words from $Ç$ having length m.

LEMMA 5.2. *For all sufficiently large numbers m,*

$$m^{(1/8)\log_\varphi m} < \operatorname{card} Ç_m < pm^{(5 + \log_\varphi m)}. \tag{72}$$

Proof. The inequality

$$\operatorname{card} Ç_m < pm^{(5 + \log_\varphi m)} \tag{73}$$

follows immediately from the representation (67) of an arbitrary word from $Ç$ and Lemma 5.1. So it suffices to prove the inequality

$$\operatorname{card} Ç_m > m^{(1/8)\log_\varphi m}. \tag{74}$$

Put

$$n = \left[\log_{4\varphi} m\right], \tag{75}$$

$$\zeta_i = \frac{m}{2(4\varphi)^i}, \ \eta_i = 2\zeta_i \qquad (i = 1, 2, \dots, n).$$

It is easy to see that

$$\eta_i > 1 \qquad (i = 1, 2, \dots n).$$

Since

$$\frac{1}{\varphi} + \frac{1}{\varphi^2} = 1,$$

we have that

$$\zeta_i > \eta_{i+1} + \eta_{i+2} \qquad (i = 1, 2, \dots, n - 2). \tag{76}$$

Let us consider the following sequence of segments:

$$\Delta_1 = [\zeta_1, \eta_1], \ \Delta_2 = [\zeta_2, \eta_2], \dots, \Delta_n = [\zeta_n, \eta_n].$$

Let

$$\alpha_1, \alpha_2, \dots, \alpha_n$$

be a finite sequence of integers such that

$$\alpha_i \in \Delta_i \qquad (i = 1, 2, \dots, n). \tag{77}$$

It is obvious that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_n,$$

and in view of (76)

$$\alpha_i > \alpha_{i+1} + \alpha_{i+2} \qquad (i = 1, 2, \dots).$$

Thus the sequence $\{\alpha_n\}$ corresponds to the negatively directed word W (67) of Fibonacci type. Clearly,

$$\sum_{i=1}^{n} \alpha_i \leq \sum_{i=1}^{n} \eta_i = \sum_{i=1}^{n} \frac{m}{(4\varphi)^i} < m \sum_{i=1}^{\infty} \frac{1}{(4\varphi)^i} = \frac{m}{4\varphi - 1} = \frac{m}{2\sqrt{5} + 1}.$$

In particular,

$$\sum_{i=1}^{n} \alpha_i < \frac{m}{5}. \tag{78}$$

Let $\mathscr{A}_m$ be the set of all possible sequences $\{\alpha_n\}$ that we have just defined.

Since $\alpha_i \in \Delta_i$ and the length of the segment Δ_i equals $m/2(4\varphi)^i$, we obtain that there exist at least $[\mathbf{m}/\mathbf{2}(\mathbf{4}\varphi)^i]$ different possibilities for α_i. Therefore,

$$\operatorname{card} \mathscr{A}_m > \prod_{i=1}^{n} \left[\frac{m}{2(4\varphi)^i} \right] > \prod_{i=1}^{n} \frac{m}{3(4\varphi)^i} = \left(\frac{m}{3} \right)^n \frac{1}{\left((4\varphi)^n\right)^{(n+1)/2}}.$$

Using the equality (75) we immediately obtain

$$\begin{aligned} \operatorname{card} \mathscr{A}_m &> \frac{3}{m} \left(\frac{1}{3} \right)^{\log_{4\varphi} m} m^{(1/2)(\log_{4\varphi} m)} \\ &> \frac{3(m^{(\log_{4\varphi} m)})}{m(m^{(\log_{4\varphi} m) + 2})^{1/2}} \cdot \frac{1}{(3^{(\log_3 m)/(\log_3 4\varphi)})} \\ &> 3(m^{(1/2)(\log_{4\varphi} m)}) \frac{1}{m^2} \cdot \frac{1}{m} = \left(\frac{3}{m^3} \right) m^{(\log_\varphi m)/(2 \log_\varphi 4\varphi)}. \end{aligned}$$

Since $\log_\varphi 4\varphi < 4$ we get that for all sufficiently large numbers m,

$$\operatorname{card} \mathscr{A}_m > m^{(1/8)\log_\varphi m}. \tag{79}$$

Let us complete the proof of our lemma. As we have noted above, every finite number sequence $\{\alpha_i\}$ satisfying the conditions (77) corresponds to some word W (67) of Fibonacci type. Now, using the inequality (78) we obtain

$$l(W) = q \sum_{i=1}^{n} \alpha_i < q \frac{m}{5}.$$

In particular, $l(W) < m$ for $q \leq 5$. Therefore

$$\operatorname{card} \mathscr{A}_m < \operatorname{card} \text{Ç}_m. \tag{80}$$

The inequality (74) now follows obviously from (79) and (80).
The lemma is proved.

Let $g_{\text{Ç}}(m)$ be the growth function of the set **Ç**. Then clearly

$$g_{\text{Ç}}(m) = \sum_{i=1}^{m} \operatorname{card} \text{Ç}_i < m \operatorname{card} \text{Ç}_m,$$

and in view of (73)

$$g_{\text{Ç}}(m) < m \cdot pm^{(5+\log_\varphi m)} = p(m^{6+\log_\varphi m}).$$

Combining (74) and the last inequality we obtain

THEOREM 5.1. *Let $[\mathbf{g}_{\text{Ç}}]$ be the growth of the set* **Ç** *of all words Fibonacci type belonging to the free semigroup F_p $(p \geq 2)$. Then*

$$[\mathbf{m}^{(1/8)\log_\varphi \mathbf{m}}] \prec [\mathbf{g}_{\text{Ç}}] \preceq [\mathbf{m}^{(\log_\varphi \mathbf{m})}].$$

In particular, the set **Ç** *has subexponential growth.*

Note that since $\ln \varphi = 0.481209\ldots$,

$$[\mathbf{m}^{0.25 \ln \mathbf{m}}] \prec [\mathbf{g}_{\text{Ç}}] \prec [\mathbf{m}^{2.07(\ln \mathbf{m})}]. \tag{81}$$

6. GROWTH AND NILPOTENCY FOR SEMIGROUPS

It is well known that the growth of every f.g. nilpotent group is a polynomial [35] and the degree of the growth can be found using Bass's formula [3]. There are several definitions of nilpotency for semigroups which state that every group which is nilpotent in the usual case would be nilpotent in the semigroup sense as well. In this section we study the growth of such nilpotent type semigroups.

The first definition of nilpotency for semigroups is due to Malcev [15]. He introduced a sequence of semigroup identities over an infinite set of variables

$$\{x, y, z_1, z_2, \dots, z_n, \dots\}$$

inductively,

$$X_o = x,\ Y_0 = y,\ X_{n+1} = X_n z_{n+1} Y_n,\ Y_{n+1} = Y_n z_{n+1} X_n, \tag{82}$$

and proved that the variety of all nilpotent groups of class c can be given by the identity $X_c = Y_c$. A semigroup satisfying this identity is called a nilpotent of class c in the sense of Malcev and is denoted by $\mathbf{M}_c$.

A similar sequence of semigroup identities,

$$p_n = q_n \qquad (n = 1, 2, \dots),$$

where

$$p_1 = xy,\ q_1 = yx,\ p_{n+1} = p_n z_{n+1} q_n,\ q_{n+1} = q_n z_{n+1} p_n, \tag{83}$$

was found by Neumann and Taylor [20]. They proved that a group is nilpotent of class c if and only if it satisfies the identity $p_c = q_c$. We will denote by $\mathfrak{B}_c$ the variety of all nilpotent semigroups of class c in the sense of Neumann−Taylor.

The third natural way to introduce the notion of nilpotency for semigroups was given by Shpilrain in [31]. He considered the set of all semigroup identities that any nilpotent group of class c satisfies. The corresponding variety is called a strictly nilpotent of class c. It follows from [15] and [20] that every cancellative nilpotent semigroup of class c can be embedded in a nilpotent group of class c. Thus every f.g. strictly nilpotent semigroup has polynomial growth.

In [36] Zimin constructed the first examples of an infinite f.g. periodic nilpotent (in the sense of Malcev and Neumann) semigroups. Sapir proved [23] that the variety of all nilpotent groups of class c cannot be defined by the system of identities over a finite set of variables even in the class of monoids for $c \geq 2$.

It was shown by Grigorchuk [11] that every cancellative semigroup of polynomial growth is almost nilpotent (in the sense of Malcev) and can be embedded in a group with polynomial growth.

In 1991 Meleshkin [16] published a paper which gave the proof that any f.g. semigroup that is a nilpotent in the sense of Malcev has a polynomial growth. However, his proof is incorrect. Moreover, the situation with the growth in this class of semigroups is quite different from the group case.

In this section we give examples of free nilpotent (in the sense of Malcev and Neumann–Taylor) f.g. semigroups having intermediate growth. The existence of free nilpotent (in the sense Neumann–Taylor) semigroups with exponential growth follows from Zimin's paper [36] (also see [30]). We note that the existence of nilpotent (in the sense Malcev) semigroups with exponential growth was obtained by M. V. Sapir (unpublished) based on his very deep results [23].

Let $\mathscr{F}_k(\mathfrak{B}_c)$ and $\mathscr{F}_k(\mathbf{M}_c)$ be respectively $\mathfrak{B}_c$-free and $\mathbf{M}_c$-free semigroups of a finite rank k.

The focus of our attention will be the element

$$p_2 \equiv xyzyx = yxzxy \equiv q_2 \tag{84}$$

of the Neumann–Taylor sequence (83).

Let $\mathbf{B} = \mathfrak{B}_2$ be a semigroup variety generated by the identity (84) and also let $\mathscr{F}_k$ and $\mathscr{M}_k$ be respectively a $\mathbf{B}$-free semigroup and a $\mathbf{B}$-free monoid of a finite rank k.

Now, we formulate one of the main results of this section.

THEOREM 6.1. *The semigroup $\mathscr{F}_k$ and the monoid $\mathscr{M}_k$ have intermediate growth for $k = 3, 4, \ldots$. The semigroup $\mathscr{F}_2$ and the monoid $\mathscr{M}_2$ have polynomial growth of degree* 4.

Proof. First of all we are going to prove the statement concerning the growth of $\mathscr{F}_2$ and $\mathscr{M}_2$. Let $\{a, b\}$ be a set of free generators for $\mathscr{F}_2$ and $\mathscr{M}_2$. It follows from the identity (84) that for any nonempty word Z over the alphabet $\{a, b\}$ the equality $baZab = abZba$ holds in both $\mathscr{F}_2$ and $\mathscr{M}_2$. Hence, using elementary transformations of the type

$$baZab \to abZba, \tag{85}$$

we get that every element $W \in \mathscr{F}_2$ (or $\mathscr{M}_2$) can be representable in one of the following forms:

(1) $\quad W = a^{\alpha} b^{\beta} ab^{\gamma} a^{\delta} \qquad (\alpha,\ \delta \geq 0,\ \beta, \gamma > 0),$

(2) $\quad W = a^{\alpha} b^{\beta} a^2 b^{\gamma} a^{\delta} \qquad (\alpha,\ \delta \geq 0,\ \beta,\ \gamma > 0),$

(3) $\quad W = a^{\alpha} b^{\beta} a^{\gamma} \qquad (\alpha, \beta, \gamma \geq 0).$

LEMMA 6.1. *Different words of the form given in* (1) *determine the different elements of the semigroup $\mathscr{F}_2$ and the monoid $\mathscr{M}_2$.*

Proof. Let G be a free nilpotent group of class 2 and rank 2 with the free generators a, b (the degree of growth of G equals 4 (see [13]). Since $G \in \mathbf{B}$ there exist the homomorphisms

$$\varphi: \mathscr{F}_2 \to G, \qquad \psi: \mathscr{M}_2 \to G$$

which are identical on the generators a, b.

It is well known (see for instance [13]) that G is isomorphic to a multiplicative group generated by two matrices

$$\bar{a} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, different words W of the type (1) correspond to the different matrices $W\varphi(W\psi)$.

This completes the proof of our lemma.

Since every element $W \in \mathscr{F}_2(\mathscr{M}_2)$ can be representable in one of the forms (1–3), by combining the preceding lemma with Propositions 2.1 and 2.2 we obtain that the degree of growth of the semigroup $\mathscr{F}_2(\mathscr{M}_2)$ is exactly equal to 4.

LEMMA 6.2. *For any natural number k the growth of the semigroup $\mathscr{F}_k$ is less than exponential.*

Proof. Let

$$a_1, a_2, \ldots, a_k \tag{86}$$

be a system of free generators for $\mathscr{F}_k$. Without loss of generality it can be assumed that $k > 1$.

It follows from the identity (84) that for an arbitrary word $U \in \mathscr{F}_{k\text{-}1}$ and for any $\gamma_1, \gamma_2, \gamma_3 > 0$,

$$a_k^{\gamma_1} U a_k^{\gamma_2} U a_k^{\gamma_3} = a_k^{\gamma_1 - 1} U a_k^{\gamma_2 + 2} U a_k^{\gamma_3 - 1}$$

in the semigroup $\mathscr{F}_k$. So, the word $a_k^{\gamma_1} U a_k^{\gamma_2} U a_k^{\gamma_3}$ is shortlex reducible relative to $\mathscr{F}_k$. Clearly, every element $W \in F_k$ can be representable in the form

$$W \equiv a_k^{\gamma_1} U_1 a_k^{\gamma_2} U_2 a_k^{\gamma_3} \cdots a_k^{\gamma_r} U_r a_k^{\gamma_r + 1}, \tag{87}$$

where $\gamma_1, \gamma_{r+1} \geq 0$, $\gamma_2, \ldots, \gamma_r > 0$, and $U_j \in \mathscr{F}_{k-1}$ $(j = 1, 2, \ldots, r)$. This yields that if W is shortlex reduced relative to $\mathscr{F}_k$, in the right-hand side of the representation (87) the number of identical factors U_j does not exceed 3.

Thus we can apply the results of Section 4. Our last remark shows that $\mathscr{F}_k = A_3(H, \mathscr{F}_{k-1})$ (see Definition 4.2) where H is a free monogenic semigroup generated by a_k. Clearly, H has linear growth. Hence, by Theorem 4.1 if the growth of $\mathscr{F}_{k-1}$ is less than exponential then the growth of $\mathscr{F}_k$ is also less than exponential. Therefore, we can finish the proof of our lemma using a simple induction.

LEMMA 6.3. *Every positively* (*negatively*) *directed word of Fibonacci type containing three different letters is an isoterm relative to the semigroup* $\mathscr{F}_3$ (*monoid* $\mathscr{M}_3$).

Proof. We shall denote the generators x_1, x_2, x_3 by the symbols a, b, and c respectively. Clearly, $\mathscr{M}_3$ is a homomorphic image of $\mathscr{F}_3$. Hence it would be sufficient to prove the statement of our lemma for the monoid $\mathscr{M}_3$.

Since $\mathscr{F}_3$ is a relatively free semigroup of the variety defined by the identity (84), $\mathscr{M}_3$ is a free monoid of the variety defined by the systems of identities

$$xyuyx = yxuxy, \qquad xy^2x = yx^2y. \tag{88}$$

Suppose that $W \in \mathscr{M}_3$ is a word of Fibonacci type (see Definition 5.2) and

$$W \equiv \prod_{j=1}^{k} (a^{\alpha_j} b^{\alpha_j} c^{\alpha_j}). \tag{89}$$

Now assume that there exists a word V over the alphabet $\{a, b, c\}$ such that V does not graphically equal W, but $W = V$ in the monoid $\mathscr{M}_3$. This implies that W contains a subword $CDUDC$ which is not graphically equal to $DCUCD$ for some (may be empty) word U. Thus the word CD is not graphically equal to DC.

Clearly,

$$W \equiv W_1 CDUDC W_2 \tag{90}$$

for some $W_1, W_2 \in M_3$.

We have to consider two cases:

(i) W is positively directed.

(ii) W is negatively directed.

Case (i). In the graphical equality (89) a finite number sequence $\{\alpha_n\}$ satisfies the inequalities (68) and (69).

Since every subword containing three different letters appears in W only once, both C and D must contain fewer than three variables. Since CD

does not graphically equal DC we obtain that at least one of the words D or C contains two different letters. Hence, we may treat only six cases:

$$C \equiv a^{\theta_1}b^{\theta_2}, D \equiv b^{\theta_3}c^{\theta_4}; \tag{1}$$

$$C \equiv a^{\theta_1}b^{\theta_2}, D \equiv c^{\theta_3}a^{\theta_4}; \tag{2}$$

$$C \equiv b^{\theta_1}c^{\theta_2}, D \equiv c^{\theta_3}a^{\theta_4}; \tag{3}$$

$$C \equiv b^{\theta_1}c^{\theta_2}, D \equiv a^{\theta_3}b^{\theta_4}; \tag{4}$$

$$C \equiv c^{\theta_1}a^{\theta_2}, D \equiv a^{\theta_3}b^{\theta_4}; \tag{5}$$

$$C \equiv c^{\theta_1}a^{\theta_2}, D \equiv b^{\theta_3}c^{\theta_4}. \tag{6}$$

Case (1). The graphical equalities (89), (90) show that

$$W \equiv W_1 a^{\theta_1}b^{\theta_2}b^{\theta_3}c^{\theta_4}Ub^{\theta_3}c^{\theta_4}a^{\theta_1}b^{\theta_2}W_2.$$

So, U is a nonempty word and

$$\alpha_j = \theta_2 + \theta_3 \geq \theta_4, \qquad \theta_4 = \alpha_{i+j},$$

for some natural numbers j and i $(i + j < k)$. Thus, we obtain $\alpha_j \geq \alpha_{i+j}$. But this contradicts the fact that W is a positive directed word.

Case (2). It follows from (89), (90) that

$$W \equiv W_1 a^{\theta_1}b^{\theta_2}c^{\theta_3}a^{\theta_4}Uc^{\theta_3}a^{\theta_4}a^{\theta_1}b^{\theta_2}.$$

Therefore $\theta_1 \leq \theta_2$ and there exists a natural number j such that $\alpha_j = \theta_2 = \theta_3$, $\theta_1 \leq \alpha_j$, $\theta_4 \leq \alpha_{j+1}$, $\theta_4 + \theta_1 = \alpha_{i+j}$ $(i \geq 2)$. Since W is a positive directed word we have that $\alpha_{i+J} \geq \alpha_{j+2}$. So, we get $\alpha_{j+2} \leq \alpha_j + \alpha_{j+1}$. This contradicts Definition 5.2 of a positively directed word of Fibonacci type.

Cases (3)–(6). These can be considered similarly.

Case (ii). Let us assume that W is a negatively directed word.

Let ψ be an antiautomorphism of a free semigroup F_3 over the alphabet $\{a, b, c\}$ which is identical on the set of generators. Clearly, for any $x, y, z \in F_3$,

$$(xyzyx)\psi = x\psi y\psi z\psi y\psi x\psi.$$

Since the semigroup $\mathscr{F}_3$ is a free semigroup of the variety defined by the identity (84) we obtain that ψ induces the antiautomorphism of the semigroup $\mathscr{F}_3$. Suppose now that $W_3 \in \mathscr{F}_3$. It is easy to see that W is a positively directed word of Fibonacci type if and only if $W\psi$ is a negatively directed word of Fibonacci type. This shows that Case ii can be reduced to Case i.

The lemma is proved.

It follows from Lemma 6.3 that different words of Fibonacci type define (in the obvious sense) the different elements of both the semigroups $\mathscr{F}_3$ and $\mathscr{M}_3$. Therefore, by Theorem 5.1 (see als (81)), we obtain

COROLLARY 6.1. *As above, let $g_{\mathscr{F}_3}(m)$, $g_{\mathscr{M}_3}(m)$ be growth functions of the semigroup $\mathscr{F}_3$ and the monoid $\mathscr{M}_3$, respectively. Then*

$$[\mathbf{g}_{\mathbf{F}_3}], [\mathbf{g}_{\mathbf{M}_3}] \succ [\mathbf{m}^{(1/4)\ln \mathbf{m}}].$$

Let us finish the proof of Theorem 6.1. By Lemma 6.2 we have that the growths of the semigroups $\mathscr{F}_3$ and $\mathscr{M}_3$ are less than exponential. It follows now from Corollary 6.1 that the growth of both $\mathscr{F}_3$ and $\mathscr{M}_3$ is greater than polynomial.

The theorem is proved.

As above, let $\mathbf{M} = \mathbf{M}_2$ be a variety of all nilpotent (in the sense of Malcev) semigroups of class nilpotency 2. Clearly the variety $\mathbf{M}$ can be defined by the identity

$$xzyz_1yzx = yzxz_1xzy. \tag{91}$$

Let $\mathscr{F}_k(\mathbf{B})$ and $\mathscr{F}_k(\mathbf{M})$ be $\mathbf{B}$-free and $\mathbf{M}$-free semigroups of finite rank k.

PROPOSITION 6.1. *Semigroup $\mathscr{F}_2(\mathbf{M})$ has polynomial growth.*

Proof. Let $\{a, b\}$ be a set of free generators for $\mathscr{F}_2(\mathbf{M})$ and $Y \in \mathscr{F}_2(\mathbf{M})$. Also, let $J_1 = b^2\mathscr{F}_2(\mathbf{M})$ and $J_2 = \mathscr{F}_2(\mathbf{M})a^2$. It follows from the equalities

$$ba^2Ya^2b = a^2bYba^2,\ b^2aYab^2 = ab^2Yb^2a,$$
$$[b(ba)ba]Y[ba(ba)b] = [ba(ba)b]Y[b(ba)ba]$$

holding in $\mathscr{F}_2(M)$ that the set of all shortlex reduced words from J_1 and J_2 have quadratic growth. Clearly, the set of all words over the alphabet $\{a, b\}$ which do not contain both a^2 and b^2 has linear growth. Hence, by Proposition 2.1 the growth function of the semigroup $\mathscr{F}_2(\mathbf{M})$ is upper bounded by some polynomial of degree 5.

THEOREM 6.2. *Semigroup $\mathscr{F}_3(\mathbf{M})$ has intermediate growth.*

Proof. Let $\mathscr{S} = \mathscr{F}_3(\mathbf{M})$ and, as above, let $\mathscr{F}_3$ be a free semigroup of the variety $\mathbf{B}$. It is easy to see that the identity (91) is a consequence of the identity (84). Thus $[\mathbf{g}_{\mathscr{S}}] \succeq [\mathbf{g}_{\mathscr{F}_3}]$ and by Theorem 6.2 the growth of $\mathscr{S}$ is greater than polynomial. Hence, it suffices to show that $\mathscr{S}$ has subexponential growth.

Let $A = \{a, b, c\}$ be a system of generators for $\mathscr{S}$. Let K_1 be the set of all words over the alphabet A which do not contain the subwords c^2 and $H_1 = \{c^\gamma : \gamma \geq 2\}$.

It is easy to see that any element $W \in \mathscr{S}$ can be representable in the form

$$W \equiv h_1 U_1 h_2 U_2 \dots h_r U_r h_{r+1}, \tag{92}$$

where

$$h_1, h_{r+1} \in H_1 \cup \{1\},$$

$h_j \in H_1$ $(j = 2, 3, \dots, r)$, $U_j \in K_1$ $(j = 1, 2, \dots, r)$, and U_j begins and ends with one of the letters a or b for $j = 2, 3, \dots, r-1$. Suppose that in the right-hand side of the representation (92) there are three identical factors $U_j \equiv U$. Then W contains a subword W_1 of the type

$$W_1 \equiv c^2 U c^2 D c^2 U c^2.$$

Applying the identity (91) for $x = c^2$, $z = U$, $y = c$, $D = z_1$, we obtain that

$$W_1 = cUc^3 Dc^3 Uc \equiv R.$$

Since a word U begins with one of the letters a or b, we have that $R \prec W_1$. Therefore, the word W_1 is shortlex reducible. Since W_1 is a subword of W, we obtain that W also is shortlex reducible. This means that $\mathscr{S}$ is a two-bounded product of the subsets H_1 and K_1 (in the sense of Definition 4.2),

$$\mathscr{S} = \mathscr{A}_2(H_1, K_1).$$

Since H_1 is a monogenic subsemigroup of $\mathscr{S}$, clearly H_1 has linear growth. By Corollary 4.1 for obtaining that $\mathscr{S}$ that subexponential growth, it suffices to prove that the subset K_1 has subexponential growth.

Let

$$H_2 = \{b^i : i \geq 2\}$$

and K_2 be the set of all words from K_1 that do not contain the subword b^2. Also, let $H_3 = \{a^i : i \geq 2\}$ and let K_3 be the set of all words from K_1 that do not contain a^2. Similarly, words of the type

$$b^2 U b^2 D b^2 U b^2, \qquad a^2 V a^2 D a^2 V a^2,$$

where U is not a power of b and V is not a power of a, are shortlex reducible relative to the semigroup $\mathscr{S}$. Hence,

$$K_1 = \mathscr{A}_2(H_2, K_2)$$

and

$$K_2 = \mathscr{A}_2(H_3, K_3).$$

Thus, again by Corollary 4.1, for our goal it suffices to prove that the set K_3 has subexponential growth.

Clearly, K_3 is the set of all elements from S that do not contain the subwords a^2, b^2, and c^2. Let T be a monogenic subsemigroup of $\mathscr{S}$ generated by the element ab and let H_4 be a set of all subwords of the words from T. It is evident that any element $V \in K_3$ can be written in the form

$$V \equiv eQ_1cQ_2cQ_3 \cdots cQ_k g, \tag{93}$$

where $e, g \in \{c\} \cup \{1\}$ and $Q_j \in H_4$ $(j = 1, 2, \ldots, k)$. It follows from (91) that the semigroup $\mathscr{S}$ satisfies the identity

$$xy^2zy^2x = y^2xzy^2x.$$

Thus, $c(ab)^2X(ab)^2c = (ab)^2cXc(ab)^2$, $c(ba)^2X(ba)^2c = (ba)^2cXc(ba)^2$. This shows that the words of the types $c(ab)^2X(ab)^2c$, $c(ba)^2X(ba)^2c$ are shortless reducible relative to $\mathscr{S}$. Therefore, in the right-hand side of (93) among the words $Q_2, Q_3, \ldots, Q_{k-1}$ there are not three different words containing $(ab)^2$ or $(ba)^2$. Let K_4 be a subsemigroup of $\mathscr{S}$ generated by the elements of the type cQ, where Q is a word over the alphabet $\{a, b\}$ that does not contain the subwords a^2, b^2, $(ab)^2$, and $(ba)^2$. Clearly, K_4 is generated by seven elements $ca, cb, cab, cba, cab, caba, cbab$, and it satisfies the identity (84). By Theorem 6.1 K_4 has subexponential growth. Hence, the graphical representation (93) shows that

$$K_3 \subseteq H_4^1 K_4 H_4^1 K_4 H_4^1 K_4 H_4^1,$$

where $H_4^1 = H_4 \cup \{1\}$, and since the set H_4 has linear growth we obtain

$$[\mathbf{g}_{K_3}] \preceq [\mathbf{m}]^4 [\mathbf{g}_{K_4}]^3 \prec [\mathbf{2^m}].$$

The theorem is proved.

7. UPPER BOUND FOR THE GROWTH OF THE SEMIGROUP $\mathscr{F}_3$

As in the previous section, Let $\mathscr{F}_3$ be a free semigroup of rank 3 of the variety defined by the identity $xyzyx = yxzxy$.

The question about the evaluation of the growth of the semigroup $\mathscr{F}_3$ was posed to the author by J. E. Pin at the NATO conference held in York

(1993). In 1990 at the international symposium in Kyoto, R. I. Grigorchuk [12] posed a problem of the existence of a cancellative semigroup whose growth is not a polynomial but is strongly smaller than the growth $[\mathbf{2}^{\sqrt{\mathbf{m}}}]$ of the function of all partitions for a natural number m. The main purpose of this section is to give a positive answer to the problem about the existence of a relatively free semigroup with the same property posed by Grigorchuk during the author's report at Professor S. I. Adian's seminar at Moscow University in May of 1995. This answer follows from the following result.

THEOREM 7.1. *The growth of the semigroup $\mathcal{F}_3$ is smaller than* $[\mathbf{m}^{\mathbf{37.8}\ln \mathbf{m}}]$.

We recall that by Corollary 5.1 the growth of the semigroup $\mathcal{F}_3$ is greater than $[\mathbf{m}^{\mathbf{1/4}\ln \mathbf{m}}]$.

Let S be a finitely generated semigroup and, as above, let $g_S(m)$ be a growth function of S.

The limit

$$\overline{\lim_{m\to\infty}} \frac{\log(\log_2 m)}{\log_2 m} = \mathbf{DIM}\ S$$

is called the *Gelfand–Kirillov superdimension of the semigroup S*.

Clearly we have

COROLLARY 7.1. *The Gelfand–Kirillov superdimension of the semigroup $\mathcal{F}_3$ is equal to zero.*

Proof of the Theorem 7.1. Let $A = \{a, b, c\}$ be a set of free generators for $\mathcal{F}_3$ and let A^+ be the set of all words over the alphabet A.

LEMMA 7.1. *Let $X \in A^+$ be a nonempty word. Then the words $baXab$, $caXac$, and $cbXbc$ are shortlex reducible relative to the semigroup $\mathcal{F}_3$.*

Proof. Indeed, it is obvious that the following equalities hold in $\mathcal{F}_3$:

$$baXab = abXba,\ caXac = acXca,\ cbXbc = bcXcb.$$

To finish the proof of the lemma it suffices to note that the words in the right-hand sides of these equalities are less (in the sense of lexicographic order) than the corresponding words on the left-hand sides.

LEMMA 7.2. *Let $W \in A^+$ and*

$$W \equiv bXaYZ,$$

where Y is nonempty and Z contains occurrences of both the words ab and ac. Then W is shortlex reducible relative to the semigroup $\mathcal{F}_3$.

Proof. Suppose that the word X is either empty or some power of the letter a. Then W contains a subword of the type $baUab$ and by Lemma 7.1

W is shortlex reducible relative to the semigroup $\mathscr{F}_3$. So we may consider only the case when X contains some occurrences of the letters b or c. Then X ends with either ba^γ or ca^γ ($\gamma \geq 0$). Therefore W contains a subword $baTab$ or $caTac$, where T is nonempty. Again by Lemma 7.1 we obtain that W cannot be a shortlex-reduced word.

The following can be proved in the analogous way.

LEMMA 7.3. *Let $W \in A^+$ and*

$$W \equiv cXaYZ$$

where (as in the preceding lemma) the word Y is nonempty and Z contains the occurrences of the words ab and ac. Then W is shortlex reducible relative to the semigroup $\mathscr{F}_3$.

By Lemmas 7.2 and 7.3 we obtain

COROLLARY 7.2. *For any $X, Y, Z \in A^+$ the words*

$$abXacYabZac,\ abXacYacZab,\ acXabYacZab,\ acZabYabZac$$

are shortlex reducible relative to the semigroup $\mathscr{F}_3$.

DEFINITION 7.1. The word W over the alphabet A is called normal if W is shortlex reduced relative to the semigroup $\mathscr{F}_3$ and it does not contain ab or ac as subwords.

LEMMA 7.4. *Any word which is shortlex reduced relative to the semigroup $\mathscr{F}_3$ is a product of 4 or fewer normal words.*

Proof. Let $W \in A^+$ be a shortlex-reduced word. Let us assume that W contains only two letters, for instance, a and b. Then W can be represented in one of the forms (1)–(3) from the proof of Theorem 6.1 and the statement of our lemma obviously takes place.

Now suppose that W contains occurrences a, b, and c and at the same time W is a product of 5 or more normal words. Then we can represent W in the form $W \equiv W_1W_2W_3W_4W_5$ where the word W_iW_{i+1} is not normal for $i = 1, 2, 3, 4$.

This means that either $W_1W_2 \equiv UabXacV$ or $W_1W_2 \equiv UacXabV$, and W_4W_5 should be a word of the type $PacYabQ$ or $PabYacQ$. Clearly, in all four possible cases we can use Corollary 7.2 for the completion of the proof of our lemma.

The lemma is proved.

Let $\mathscr{G}$ be a set of all normal words. Also, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be the sets of all normal words which contain ab (respectively ac) and do not contain ac (respectively ab) as subwords. Also, let $\mathscr{G}_3 = \mathscr{G} \setminus (\mathscr{G}_1 \cup \mathscr{G}_2)$. Since $\mathscr{G} = \mathscr{G}_1$

$\cup\, \mathscr{G}_2 \cup \mathscr{G}_3$ we obtain that

$$g_{\mathscr{G}} = g_{\mathscr{G}_1} + g_{\mathscr{G}_2} + g_{\mathscr{G}_3}. \tag{94}$$

LEMMA 7.5. *The set $\mathscr{G}_3$ has polynomial growth of degree* 5.

Proof. Indeed, let $W \in \mathscr{G}_3$. Then W is a shortlex-reduced word which does not contain ab and ac as subwords. This means that W can be represented in the form

$$W \equiv W_1 a^{\alpha} \ (\alpha \geq 0), \tag{95}$$

where W_1 is a word over the alphabet $\{b, c\}$. It follows from Theorem 6.1 that a subsemigroup of $\mathscr{F}_3$ generated by b and c has polynomial growth of degree 4. Therefore (95) shows that

$$[\mathbf{g}_{\mathbf{G}_3}] = [\mathbf{m}^4][\mathbf{m}] = [\mathbf{m}^5]. \tag{96}$$

The lemma is proved.

Now, we are going to get some information about the upper bound for the growth of the sets $\mathscr{G}_1$ and $\mathscr{G}_2$.

DEFINITION 7.2. The word $W \in \mathscr{G}_1$ ($\mathscr{G}_2$) is called perfect of type 1 (respectively, type 2) if W does not contain the subword cb (respectively, bc).

In particular (see Definition 7.1), every perfect word is shortlex reduced relative to the semigroup $\mathscr{F}_3$.

DEFINITION 7.3. The perfect word of the form

$$V \equiv \prod_{i=1}^{k} a^{\alpha_i} b^{\beta_i} c^{\gamma_i}.$$

is called an *integer word of type* 1. Each factor $a^{\alpha_i} b^{\beta_i} c^{\gamma_i}$ is called a block of V. Dually, the word of the form

$$R \equiv \prod_{i=1}^{k} c^{\gamma_i} b^{\beta_i} a^{\alpha_i}$$

is said to be an *integer word of type* 2 and its factors $c^{\gamma_i} b^{\beta_i} a^{\alpha_i}$ are called the integer blocks, or for short the blocks, of R.

DEFINITION 7.4. As above, let W be a word over the alphabet $A = \{a, b, c\}$. The minimal number q such that W is a product of q powers of generators is called the height of W relative to A, or for short the height of W, and is denoted by $h(W)$.

It is easy to see that the heights of both V and R relative to the set A are equal to $3k$.

The following lemma follows easily from our definitions.

LEMMA 7.6. *Let $W \in \mathscr{F}_3$. Let us also assume that W contains two integer subwords of types* 1 *and* 2. *If each of these subwords has more than one block, then W is shortlex reducible relative to the semigroup $\mathscr{F}_3$.*

LEMMA 7.7. *Every normal word of height greater than* 5 *contains a subword of an integer word of type* 1 *or* 2 *whose height equals* 3.

Proof. Let W be a word satisfying the condition of our lemma. By Lemma 7.5 $W \notin \mathscr{G}_3$. Thus either $W \in \mathscr{G}_1$ or $W \in \mathscr{G}_2$. Suppose that $W \in \mathscr{G}_1$. Then W contains a subword ab and at the same time W does not contain any occurrences of ac.

Suppose that W does not contain the letter c. Then W can be representable in one of the forms (1–3) from the proof of Theorem 6.1. This contradicts the fact that $h(W) > 5$. Thus W contains the letter c. This means that W contains a subword of one of the following types:

$$a^{\alpha} b^{\beta} c^{\gamma},\ b^{\beta} c^{\gamma} a^{\alpha},\ c^{\gamma} a^{\alpha} b^{\beta}.$$

Clearly all these words are the subwords of some integer perfect words of type 1.

The case when $W \in \mathscr{G}_2$ is analogous.

The lemma is proved.

LEMMA 7.8. *Every word $W \in \mathscr{G}_1$ can be written in the form*

$$W \equiv W_1 g, \tag{97}$$

where W_1 is a subword of an integer perfect word of type 1 *and $h(g) \le 8$.*

Proof. Suppose to the contrary. Then in the representation (97) $h(g) \le 8$ and at the same time W_1 contains the subword ba or cb. Let $\overline{W}$ be the maximal prefix of W that does not contain both ba and cb. Then either

$$W \equiv \overline{W} ba D_1 \tag{98}$$

or

$$W \equiv \overline{W} cb D_2, \tag{99}$$

where $D_1, D_2 \in \mathscr{G}_1$ are some normal words. Clearly $\overline{W}$, as does W, belongs to the set $\mathscr{G}_1$.

Let us assume that the graphical equality (98) takes place. If D_1 is empty, then $W \equiv \overline{W} ba$ and the statement of our lemma holds.

Suppose now that D_1 is nonempty and let t_1 be the initial letter of the word D_1. Then $D_1 \equiv t_1 D_1'$ for some word D_1'. Since $\mathrm{ba} t_1 D_1'$ is a subword of the shortlex-reduced word W, by Lemma 7.1 the word D_1' does not contain the subword ab. At the same time D_1' is a subword of W which does not contain the subword ac. This means that D_1' can be represented in the form $D_1' \equiv C_1(b,c)a^\gamma$, where C_1 is a word over the alphabet $\{b,c\}$. So, we may consider C_1 as an element of the 2-generated **B**-free semigroup which is isomorphic to $\mathscr{F}_2$. It follows from the proof of Theorem 6.1 that $h(C_1) \le 4$. Thus the height of the word $baD_1 \equiv \mathrm{bat}_1 C_1(b,c)a^\gamma$ is less than or equal to 8. Now putting in the right-hand side of (97) $W_1 \equiv \overline{W}$ and $h \equiv baD_1$, we obtain the statement of our lemma.

The case of the graphical equality (98) is analogous.

The lemma is proved.

The following lemma can be proved using the same arguments.

LEMMA 7.9. *Any word $V \in \mathscr{G}_2$ can be written in the form*

$$V \equiv h_1 V_1,$$

where V_1 is a subword of an integer perfect word of type 2 and the height of h_1 is less than or equal to 8.

LEMMA 7.10. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the sets of all integer perfect words of types 1 and 2 respectively and, as above, let $\mathscr{G}$ be the set of all normal words over the alphabet A. Then*

$$[\mathbf{g}_{\mathscr{G}}] \preceq \big([\mathbf{g}_{\mathscr{P}_1}] + [\mathbf{g}_{\mathscr{P}_2}]\big)[\mathbf{m}^8]. \tag{100}$$

Proof. Indeed, by Lemmas 7.8 and 7.9 we have

$$[\mathbf{g}_{\mathscr{G}_1}] \preceq [\mathbf{g}_{\mathscr{P}_1}][\mathbf{m}^8], \qquad [\mathbf{g}_{\mathscr{G}_2}] \preceq [\mathbf{g}_{\mathscr{P}_2}][\mathbf{m}^8]. \tag{101}$$

Combining Lemmas 7.5 and (101) with the equality (94), we immediately obtain the relation (100).

The lemma is proved.

It follows from Lemmas 7.4 and 7.10 that every upper bound for the growth of the sets $\mathscr{P}_1$ and $\mathscr{P}_2$ produces some upper bound for the growth of the semigroup $\mathscr{F}_3$.

LEMMA 7.11. *As above, let $\mathscr{P}_1, \mathscr{P}_2$ be the sets of all integer perfect words of types 1 and 2 respectively. Then*

$$[\mathbf{g}_{\mathscr{P}_2}] \prec [\mathbf{g}_{\mathscr{P}_1}][\mathbf{m}^{12}].$$

Proof. Let W be a perfect word of type 2. Let us also assume that $h(W) > 12$. Since the height of each integer block equals 3, W has a prefix containing exactly two integer blocs. Let W_1 be a maximal prefix of W

containing the first integer block. In particular, $h(W_1) \le 5$. The condition on the height of W shows that this word can be written in the form

$$W \equiv W_1 c^{\gamma} b^{\beta} a^{\alpha} V c^{\gamma_1} b^{\beta_1} a^{\alpha_1} W_2, \tag{102}$$

where V is an integer word and $1 \le h(W_2) \le 3$. We say that V is a *q-subword* of W. Clearly, W_2 begins with a letter c.

Let $\overline{V}$ be an arbitrary word over alphabet A that equals V in the semigroup $\mathscr{F}_3$. First note that W is shortlex reduced relative to $\mathscr{F}_3$ and that W_1 contains the subwords ba and cb. Thus, by Lemma 7.1, $\overline{V}$ does not contain the subwords ab and bc. Since the suffix aW_2 of W begins with the subword ac, in addition V does not contain the subword ca.

Let ψ be an antiautomorphism of the absolutely free semigroup F_3 over the alphabet A, that is, identical on the set of generators $\{a, b, c\}$, and $\preceq$ be a lexicographic order on the set A^+. As we have already mentioned in the proof of Lemma 6.3, ψ induces the automorphism of the **B**-free semigroup $\mathscr{F}_3$. It follows easily from the previous restrictions on $\overline{V}$ that every word $R \in A^+$ such that $R = \overline{V}\psi$ in $\mathscr{F}_3$ must not contain the subwords ba, cb, ac.

This means that R is equal to a perfect word of type 1. Therefore, the set of all q-subwords of all integer perfect words of type 2 having length m is in one-to-one correspondence with some subset of the set of all integer perfect words of type 1 having the same length m. Thus, using the representation (102) we immediately obtain the statement of our lemma. The lemma is proved.

LEMMA 7.12. *The semigroup $\mathscr{F}_3$ and the set $\mathscr{P}_1$ of all integer perfect words of type* 1 *have the same growth. In other words,*

$$[\mathbf{g}_{\mathscr{F}_3}] = [\mathbf{g}_{\mathscr{P}_1}]. \tag{103}$$

Proof. Indeed, suppose that $W \in A^+$ and that the word W is shortlex reduced relative to $\mathscr{F}_3$. In addition, suppose that W contains integer perfect subwords of types 1 and 2 with heights greater than 3. Clearly, every subword of an integer perfect word with height greater than 9 contains at least two blocks. Thus by Lemma 7.6 the word W does not contain any integer perfect subwords of types 1 and 2 with height greater than 9. It is easy to see that if $W \equiv XUYVZ$ where U and V are integer perfect words of the same type and $h(U), h(V) > 9$ then UYV is an integer perfect word.

On the other hand, by Lemma 7.4 every word W which is shortlex reduced (relative to $\mathscr{F}_3$) is graphically equal to the product of 4 or fewer normal subwords. Hence without loss of generality we may assume that in the given graphical representation of W the number of different normal factors with height greater than 9 is never greater than 3. Combining these results with Lemmas 7.7, 7.10, and 7.11 we obtain

$$[\mathbf{g}_{\mathbf{F}_3}] \preceq [\mathbf{g}_{\mathscr{P}_1}][\mathbf{m}^{12}] \cdot [\mathbf{m}]^8 \cdot [\mathbf{m}^{10}]^3 = \left[\mathbf{g}_{\mathscr{P}_1}\mathbf{m}^{50}\right].$$

At the same time, it is obvious that the set $\mathscr{P}_1$ contains the set of all positively directed words of Fibonacci type. So, by Lemma 5.2, $g_{\mathscr{P}_1}(m) > m^{(1/4)\ln m}$ for all sufficiently large numbers m. This yields that

$$g_{\mathscr{P}_1}(m) = m^{\varsigma(m)},$$

where $\varsigma(m)$ is a monotone increasing function satisfying the inequality

$$\varsigma(m) > \frac{1}{4}\ln m.$$

Thus, $[\mathbf{g}_{\mathscr{P}_1}][\mathbf{m}^{50}] = [\mathbf{g}_{\mathscr{P}_1}]$. So, $[\mathbf{g}_{\mathscr{F}_3}] \preceq [\mathbf{g}_{\mathscr{P}_1}]$ and in view of $\mathscr{P}_1 \subset \mathscr{F}_3$ we get the equality (103).

The lemma is proved.

LEMMA 7.13. *Let U be a perfect word of type* 1. *Then U does not contain any subwords of the type*

$$U_1 \equiv Xb^{\beta}c^{\gamma}a^{\alpha}Yc^{\gamma}a^{\alpha}b^{\delta}c \tag{104}$$

where Y is nonempty and X ends with the letter a and contains at least two integer blocs.

Proof. Suppose to the contrary. First note that X as a subword of U is also a perfect word of type 1. Since X contains two integer blocks,

$$X \equiv X_1caX_2a \tag{105}$$

for some nonempty word X_2.

Let us assume that in (104) $\beta \le \delta$. Then

$$U_1 \equiv X(b^{\beta}c^{\gamma}a^{\alpha})Y(c^{\gamma}a^{\alpha}b^{\beta})b^{\delta-\beta}c = X(c^{\gamma}a^{\alpha}b^{\beta})Y(b^{\beta}c^{\gamma}a^{\alpha})b^{\delta-\beta}c \equiv V_1 \tag{106}$$

in the semigroup $\mathscr{F}_3$. It follows from the graphical equalities (105), (106) that the word V_1 begins with the subword

$$Z \equiv Xc \equiv X_1caX_2ac. \tag{107}$$

Obviously $Z = X_1acX_2ca$ in the semigroup $\mathscr{F}_3$. Therefore

$$V_1 = X_1acX_2caH \equiv V_2 \tag{108}$$

for some $H \in A^+$. Put

$$D_1 \equiv X_1 ca X_2, \qquad D_2 \equiv X_1 ac X_2.$$

In view of the graphical equalities (106)–(108), D_1 and D_2 are the prefixes of the words U_1 and V_2 respectively. At the same time, clearly $U_1 = V_2$ in $\mathscr{F}_3$ and

$$D_2 \prec D_1.$$

Thus, U_1 is shortlex reducible relative to $\mathscr{F}_3$. This contradicts the fact that U_1 is a subword of the perfect word U (which is shortlex reduced relative to $\mathscr{F}_3$ by the definition).

Let us assume now that in (104) $\beta > \delta$.

Then

$$\begin{aligned} U_1 &\equiv X b^{\beta-\delta}(b^\delta c^\gamma a^\alpha) Y (c^\gamma a^\alpha b^\delta) c \\ &= X b^{\beta-\delta}(c^\gamma a^\alpha b^\delta) Y (b^\delta c^\gamma a^\alpha) c \\ &\equiv X b^{\beta-\delta} c^{\gamma-1} ca (a^{\alpha-1} b^\delta Y b^\delta c^\gamma a^{\alpha-1}) ac \\ &= X b^{\beta-\delta} c^{\gamma-1} ac (a^{\alpha-1} b^\delta Y b^\delta c^\gamma a^{\alpha-1}) ca \\ &\equiv XTacR, \end{aligned}$$

where

$$T \equiv b^{\beta-\delta} c^{\gamma-1} \qquad \text{and} \qquad R \equiv a^{\alpha-1} b^\delta Y b^\delta c^\gamma a^{\alpha-1} ca.$$

Clearly we can rewrite (105) in the form $X \equiv X_1 ca \overline{X}_2$ (where $\overline{X}_2 = X_2 a$). So, $U_1 \equiv \overline{X}_1 ca \overline{X}_2 TacR = \overline{X}_1 ac \overline{X}_2 TcaR \equiv U_3$. It is easy to see that $U_3 \prec U_1$. Since U_1 is a subword of U, we obtain that the word U is shortlex reducible. We obtain the same contradiction as in the previous case.

The lemma is proved.

Let

$$W \equiv \prod_{i=1}^{k} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} \tag{109}$$

be an integer perfect word of type 1. Also, let $1 \le i < j \le k$.

LEMMA 7.14. *For an arbitrary perfect word W (109) the implications*

$$\alpha_i \le \alpha_j \Rightarrow \beta_i < \beta_j \qquad (i > 1), \tag{110}$$

$$\beta_i > \beta_j \Rightarrow \gamma_i > \gamma_j \qquad (j - i > 1) \tag{111}$$

hold. If in the right-hand side of (109) $j < k$, *then*

$$\gamma_i \ge \gamma_j \Rightarrow \alpha_{i+1} > \alpha_{j+1} \qquad (j - i > 1). \tag{112}$$

If, in addition, $\alpha_{i+1} \neq 1, 2$ *then*

$$\beta_i > \beta_{i+1} \Rightarrow \gamma_i > \gamma_{i+1}. \tag{113}$$

If $\beta_{i+1} \neq 1, 2$ *then*

$$\gamma_i \geq \gamma_{i+1} \Rightarrow \alpha_{i+1} > \alpha_{i+2}. \tag{114}$$

Proof. Let us prove the implication (110). Suppose that in the right-hand side of the graphical equality (109) $\alpha_i \leq \alpha_j$ and at the same time $\beta_i \geq \beta_j$. Since $i > 1$, W can be written in the form

$$W \equiv W_1 c a^{\alpha_i} b^{\beta_j} W_2 a^{\alpha_i} b^{\beta_j} c W_3, \tag{115}$$

where the word W_2 begins with a subword $b^{\beta_i - \beta_j}$ and ends with a subword $a^{\alpha j - \alpha_i}$. It is obvious that W_2 contains an occurrence of the letter c. In particular, the word W_2 is not empty. Since the semigroup $\mathscr{F}_3$ satisfies the identity $xyzyx = yxzxy$, by putting

$$x = c,\ y = a^{\alpha_i} b^{\beta_j},\ z = W_2$$

we obtain that the equality

$$c a^{\alpha_i} b^{\beta_j} W_2 a^{\alpha_i} b^{\beta_j} c = a^{\alpha_i} b^{\beta_j} c W_2 c a^{\alpha_i} b^{\beta_j}$$

holds in the semigroup $\mathscr{F}_3$. This implies that

$$W = W_1 a^{\alpha_i} b^{\beta_j} c W_2 c a^{\alpha_i} b^{\beta_j} W_3 \equiv V$$

in $\mathscr{F}_3$. Combining the last equality with the graphical equality (115) we obtain $V \prec W$. This contradicts the fact that the word W is shortlex reduced relative to the semigroup $\mathscr{F}_3$.

The implication (110) is proved.

Now let us prove the implications (111) and (113). Suppose that in the right-hand side of the graphical equality (109) $\beta_i > \beta_j$ for some integers i, j, and at the same time $\gamma_i \leq \gamma_j$. Then

$$b^{\beta_i} c^{\gamma_i} \equiv b^{\beta_i - \beta_j} b^{\beta_j} c^{\gamma_i}, \qquad b^{\beta_j} c^{\gamma_j} \equiv b^{\beta_j} c^{\gamma_i} c^{\gamma_j - \gamma_i}.$$

If in addition $j - i > 1$ or $i = j + 1$ and $\alpha_{i+1} > 2$, W has the form

$$W \equiv W_1 b^{\beta_j} c^{\gamma_i} a W_2 a b^{\beta_j} c^{\gamma_i} W_3$$

where a word W_2 is nonempty. Therefore,

$$W = W_1 a b^{\beta_j} c^{\gamma_i} W_2 b^{\beta_j} c^{\gamma_i} a W_3 \equiv V$$

in the semigroup $\mathcal{F}_3$. It is easy to see that $V \prec W$. We again obtain the contradiction with the fact that the word W is shortlex reduced relative to $\mathcal{F}_3$.

The implications (111), (113) are proved.

The final part of the statement of our lemma about the implications (112) and (114) can be proved in an analogous way.

The lemma is proved.

LEMMA 7.15. *As above, let W (109) be an integer perfect word of type 1. Also, let $k > 4$ in the graphical equality (109). Then for every $i \geq 3$ and for an arbitrary natural number j such that $i + 1 < j \leq k$, the implication*

$$\alpha_i > \alpha_j \Rightarrow \beta_i > \beta_j \tag{116}$$

holds. If, in addition, in the right-hand side of the graphical equality (109) $\gamma_i \neq 1, 2$, then the implication

$$\alpha_i > \alpha_{i+1} \Rightarrow \beta_i > \beta_{i+1} \tag{117}$$

holds.

Proof. Suppose otherwise. Then there exist some integers i and j satisfying the conditions of our lemma such that

$$\alpha_i > \alpha_j \tag{118}$$

and at the same time

$$\beta_i \leq \beta_j. \tag{119}$$

Since in the right-hand side of (109) $i \geq 3$ and $k > 4$, this graphical equality can be rewritten in the form

$$W \equiv W_0 ca W_1 a^{\alpha_i} b^{\beta_i} c W_2 ca^{\alpha_j} b^{\beta_j} W_3 \tag{120}$$

where W_1 contains at least one integer block. In particular, $l(W_1) > 1$.

It is easy to see that if $j > i + 1$ or $\gamma_i > 2$, the word W_2 is also nonempty. It follows from the inequalities (118), (119) that in both cases

$$\begin{aligned} W &\equiv W_0 ca W_1 a^{\alpha_i - \alpha_j} (a^{\alpha_j} b^{\beta_i} c) W_2 (ca^{\alpha_j} b^{\beta_i}) b^{\beta_j - \beta_i} W_3 \\ &= W_0 ca W_1 a^{\alpha_i - \alpha_j} (ca^{\alpha_j} b^{\beta_i}) W_2 (a^{\alpha_j} b^{\beta_i} c) b^{\beta_j - \beta_i} W_3 \end{aligned}$$

in the semigroup $\mathcal{F}_3$.

Therefore, we have the equality of the type

$$W = W_0 ca \overline{W}_1 ac R$$

where $\overline{W}_1 \equiv W_1 a^{\alpha_i - \alpha_j - 1}$ and so $l(\overline{W}_1) \geq l(W_1) > 1$. This shows that $\overline{W}_1$ is a nonempty word and therefore

$$W = W_0 ac\overline{W}_1 caR \tag{121}$$

in the semigroup $\mathscr{F}_3$. A simple comparison of (120) and (121) shows that the word W is shortlex reducible relative to the semigroup $\mathscr{F}_3$. This contradicts the fact that W is shortlex reduced relative to $\mathscr{F}_3$. Hence the implications (116), (117) hold.

This completes the proof of our lemma.

Now put

$$A_1 \equiv abc,\ A_2 \equiv bca,\ A_3 \equiv cab,\ A_4 \equiv ab^2c,\ A_5 \equiv bc^2a,\ A_6 = ca^2b. \tag{122}$$

Clearly, if the word W (109) does not contain the subwords (122) then in the right-hand side of (109) $\alpha_i\ \beta_i,\ \gamma_i \neq 1, 2$ for $1 < i < k$.

The following two lemmas can be proved using the same arguments as in the proofs of Lemmas 7.14 and 7.15.

LEMMA 7.16. *Let W (109) be an integer perfect word of type 1 containing more than four blocs. Also, do not let W contain the subwords (122). Then for any integer $i > 4$ the implications*

$$\alpha_i > \alpha_{i+1} \Rightarrow \beta_i > \beta_{i+1},$$
$$\beta_i > \beta_{i+1} \Rightarrow \gamma_i > \gamma_{i+1},$$
$$\gamma_i > \gamma_{i+1} \Rightarrow \alpha_{i+1} > \alpha_{i+2}$$

take place. In particular,

$$\alpha_i > \alpha_{i+1} \Rightarrow \alpha_{i+1} > \alpha_{i+2},$$
$$\beta_i > \beta_{i+1} \Rightarrow \beta_{i+1} > \beta_{i+2},$$
$$\gamma_i > \gamma_{i+1} \Rightarrow \gamma_{i+1} > \gamma_{i+2}.$$

LEMMA 7.17. *Let W be a perfect word satisfying the conditions of the previous lemma. Also, let i, j be integers such that $i < j$. If $i \geq 4$ then the implications*

$$\alpha_i \leq \alpha_j \Rightarrow \gamma_{i-1} < \gamma_{j-1},$$
$$\gamma_{i-1} < \gamma_{j-1} \Rightarrow \beta_{i-1} < \beta_{j-1},$$
$$\beta_{i-1} < \beta_{j-1} \Rightarrow \alpha_{i-1} \leq \alpha_{j-1}$$

hold. In particular,

$$\alpha_i \le \alpha_j \Rightarrow \alpha_{i-1} \le \alpha_{j-1},$$
$$\beta_i < \beta_j \Rightarrow \beta_{i-1} < \beta_{j-1},$$
$$\gamma_i < \gamma_j \Rightarrow \gamma_{i-1} < \gamma_{j-1}.$$

LEMMA 7.18. *Let W be an integer perfect word of type* 1 *given by the graphical representation* (109). *Then in the right-hand side of* (109),

$$\alpha_i \ne \alpha_{i+q},\ \beta_i \ne \beta_{i+q},\ \gamma_i \ne \gamma_{i+q} \qquad (\textit{for } i = 4,5,\dots,\ q = 2,3,\dots). \tag{123}$$

If in addition W does not contain occurrences of the words (122) *then*

$$\alpha_i \ne \alpha_{i+1},\ \beta_i \ne \beta_{i+1},\ \gamma_i \ne \gamma_{i+1} \qquad (\textit{for } i = 4,5,\dots). \tag{124}$$

Proof. Suppose to the contrary and let

$$\alpha_i = \alpha_{i+q} \tag{125}$$

for some $i \ge 4$ and $q \ge 1$. First note that, by Lemma 7.17, in the right-hand side of the graphical equality (109) there should be

$$\gamma_{i-1} < \gamma_{i+q-1}.$$

Therefore

$$c^{\gamma_{i+q-1}} a^{\alpha_{i+q}} b^{\beta_{i+q}} \equiv \left(c_1^{\gamma_{i+q-1}-\gamma_{i-1}}\right)\left(c^{\gamma_{i-1}} a^{\alpha_i} b^{\beta_i+q}\right). \tag{126}$$

Now the graphical equalities (109) and (126) show that

$$W \equiv X b^{\beta_{i-1}} c^{\gamma_{i-1}} a^{\alpha_i} Y c^{\gamma_{i-1}} a^{\alpha_i} b^{\beta_{i+q}} Z \tag{127}$$

for some nonempty words X, Y, Z.

Since W is a perfect word and $i \ge 4$, the word X contains at least two integer blocs and ends with the letter a. Thus, by Lemma 7.13 we obtain that the word W is shortlex reducible relative to the semigroup $\mathscr{F}_3$. This contradiction shows that the equality (125) does not hold.

The inequalities $\beta_i \ne \beta_{i+q}$ from (123) and $\beta_i \ne \beta_{i+1}$ from (124) follow easily from Lemmas 7.14 and 7.15 (see the implications (110) and (116), (117)).

Finally, let us assume that in (109)

$$\gamma_i = \gamma_{i+q}.$$

Put in the condition of Lemma 7.16 that $j = i + q$. In view of Lemma 7.14 we get that in the right-hand side of (109)

$$\alpha_{i+1} > \alpha_{i+q+1}.$$

Then

$$c^{\gamma_i} a^{\alpha_{i+1}} \equiv c^{\gamma_i} a^{\alpha_{i+q+1}} a^{\alpha_{i+1} - \alpha_{i+q+1}},$$

and at the same time, by our assumption,

$$c^{\gamma_{i+q}} a^{\alpha_{i+q+1}} \equiv c^{\gamma_i} a^{\alpha_{i+q+1}}.$$

Hence, we have

$$W \equiv X b^{\beta_i} c^{\gamma_i} a^{\alpha_{i+q+1}} Y c^{\gamma_i} a^{\alpha_{i+q+1}} b^{\beta_{i+q+1}} Z$$

for some X, Y, Z as in the condition of Lemma 7.13. Applying this lemma we obtain that the word W can't be perfect, so that one of the conditions of our lemma fails.

The lemma is proved.

LEMMA 7.19. *Let W be an integer perfect word of type 1 with $k > 5$ blocks given by the graphical representation* (109). *Also, let $\overline{W}$ be a suffix of W containing fewer than $k - 3$ blocks and $V \in A^+$ be a word such that $V = \overline{W}$ in the semigroup $\mathscr{F}_3$. Then V does not contain a subword ac.*

Proof. Suppose the contrary and let the equality of the type

$$\overline{W} = V \equiv X_1 ac X_2 \tag{128}$$

hold in the semigroup $\mathscr{F}_3$.

Clearly, we can rewrite the graphical equality (109) in the form

$$W \equiv a^{\alpha_1} b^{\beta_1} c^{\gamma_1 - 1} ca Q \overline{W}$$

where $Q \in A^+$ is a nonempty word. It follows from (128) that the equality

$$W = a^{\alpha_1} b^{\beta_1} c^{\gamma_1 - 1} ca Q X_1 ac X_2 \equiv \hat{W}$$

also holds in $\mathscr{F}_3$.

Let W_1 and $\hat{W}_1$ be the beginnings of the length $\alpha_1 + \beta_1 + \gamma_1$ for the words W and $\hat{W}_1$ respectively. Obviously, $W_1 \equiv a^{\alpha_1} b^{\beta_1} c^{\gamma_1}$ and $\hat{W}_1 \equiv a^{\alpha_1} b^{\beta_1} c^{\gamma_1 - 1} a$. Since $\hat{W}_1 \prec W_1$, we get $\hat{W} \prec W$. This contradicts the fact that W is a shortlex-reduced word.

The lemma is proved.

LEMMA 7.20. *Let W be an integer perfect word of type 1 with k $(k > 5)$ blocks given by the graphical equality* (109) *and let*

$$\overline{W} \equiv \prod_{i=5}^{k} a^{\alpha_i} b^{\beta_i} c^{\gamma_i} \tag{129}$$

be the suffix of W having $k-5$ *blocks. Then* $\overline{W}$ *does not contain subwords of the type* A_iYA_j *with* A_i, A_j *from* (122), $i, j \in \{1,2,3\}$, *and the height of the word* Y *greater than or equal to* 3.

Proof. Since $h(Y) \geq 3$, in view of Lemma 7.18 we may consider only the case when $i \neq j$. Since $A_1 \prec A_2 \prec A_3$ and the equalities

$$A_2YA_1 \equiv (bc)aYa(bc) = a(bc)Y(bc)a \equiv A_1YA_2,$$
$$A_3YA_1 \equiv c(ab)Y(ab)c = (ab)cYc(ab) \equiv A_1YA_3,$$
$$A_3YA_2 \equiv (ca)bYb(ca) = b(ca)Y(ca)b \equiv A_2YA_3$$

hold in the semigroup $\mathscr{F}_3$, without loss of generality we may treat only the following cases:

(a) $\overline{W} \equiv XA_1YA_2$,
(b) $\overline{W} \equiv XA_1YA_3$,
(c) $\overline{W} \equiv XA_2YA_3Z$
$(X, Z \in A^+)$.

Case (a). Taking into account that $\overline{W}$ is a subword of the word W (109), Y begins with one of the letters a or c.

Let us assume that $Y \equiv aY_1$, then $\overline{W}$ contains the subword $abcaY_1bca$. Since $h(Y) \geq 3$, the word Y_1 contains the letters b and c. Then in the graphical equality (109) $\gamma_i = \gamma_{i+q}$ for some number $q \geq 2$. This contradicts Lemma 7.18. Thus Y begins with a letter c and $Y \equiv cY'$ $(Y' \in A^+)$. Then

$$\overline{W} \equiv Xa(bc)cY'(bc)aZ = X(bc)acY'a(bc)Z \equiv V \qquad (Z \in A^+).$$

Since V contains a subword ac, we get a contradiction with Lemma 7.19.

Case (b). Obviously $\overline{W} \equiv XabcYcabZ$, where X ends with the letter a or c. Let us assume that $X \equiv X'a$. Then

$$\overline{W} \equiv X'a(abc)Y(cab)Z = X'a(cab)Y(abc)Z \equiv V',$$

and since V' contains a subword ac we get a contradiction with Lemma 7.19. Suppose now that $X \equiv X''c$. Then $\overline{W} \equiv X''cabcYcabZ$. Since $h(Y) > 3$ we obtain that in the graphical representation (109) $\alpha_{i+q} = \alpha_i$ for some number $q \geq 2$. This contradicts Lemma 7.18.

Case (c). This case contradicts the conclusion of Lemma 7.13 for $\alpha = \beta = \gamma = 1$.

The lemma is proved.

The following lemma can be proved using the same methods.

LEMMA 7.21. *As above, let W be an integer perfect word of type* 1 *given by the equality* (109) *with k ($k > 5$) integer blocks. Also, let $\overline{W}$* (129) *be the suffix of W having $k - 5$ blocks. Then $\overline{W}$ does not contain subwords of the type A_iYA_j with A_i, A_j from* (122), $i, j \in \{4, 5, 6\}$, *and* $h(Y) \geq 3$.

DEFINITION 7.5. Let W be an integer perfect word of type 1. We say that W is special if it does not contain the subwords A_q ($1 \leq q \leq 6$) from (122).

Let $\mathscr{Q}$ be a set of all special words.

The following result follows obviously from Lemmas 7.20 and 7.21.

LEMMA 7.22. *As above, let W be an integer perfect word* (109) *of type* 1 *having $k > 5$ blocs and let $\overline{W}$* (129) *be the suffix of W containing $k - 5$ blocs. Then $\overline{W}$ can be representable in one of the forms*

$$R_1,\ R_1A_iR_2,\ R_1A_iR_2A_jR_3, \tag{130}$$

where R_1, R_2, and R_3 are special words or empty symbols.

Clearly every integer perfect word containing 5 or fewer blocks has height ≤ 15. Thus, taking into account Lemma 7.22, the graphical representation (129), and the fact that the growth of the set of all words of type (130) is $[(\mathbf{g}_{\mathscr{Q}})^3]$, we obtain $[\mathbf{g}_{\mathscr{P}_1}] \preceq [\mathbf{m}^{15}][(\mathbf{g}_{\mathscr{Q}})^3]$. It was noted in the proof of Lemma 7.12 that $[\mathbf{g}_{\mathscr{P}_1}] \succeq [\mathbf{m}^{(1/4)\ln \mathbf{m}}]$. Then $[(\mathbf{g}_{\mathscr{Q}})^3] \succeq [\mathbf{m}^{(1/4)\ln \mathbf{m}}]$ and so $[\mathbf{m}^{15}][(\mathbf{g}_{\mathscr{Q}})^3] = [(\mathbf{g}_{\mathscr{Q}})^3]$. Hence we have

COROLLARY 7.3. *The growth and $[\mathbf{g}_{\mathscr{Q}}]$ of the set $\mathscr{Q}$ and the growth $[\mathbf{g}_{\mathscr{P}_1}]$ of the set of all perfect words of type* 1 *are related by*

$$[\mathbf{g}_{\mathscr{P}_1}] \preceq \left[(\mathbf{g}_{\mathscr{Q}})^3\right].$$

DEFINITION 7.6. A special word W given by the equality (109) is called positively (negatively) directed if in the right-hand side of (109)

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{k-1} < \alpha_k$$

(respectively, $\alpha_1 > \alpha_2 > \cdots > \alpha_{k-1} > \alpha_k$).

The following lemma shows the structure of the special words.

LEMMA 7.23. *Let W* (109) *be a special word with $k > 5$ blocks. Then the suffix $\overline{W}$* (129) *of the word W satisfies one of the following three conditions*:

(a) *$\overline{W}$ is a positively directed word*,

(b) *$\overline{W}$ is a negatively directed word*, *or*

(c) *$\overline{W}$ is a product of positively and negatively directed words and the word whose height is less than or equal to* 3.

Proof. Indeed, if $\alpha_5 > \alpha_6$ then by Lemma 7.16

$$\alpha_5 > \alpha_6 > \alpha_7 > \alpha_8 > \cdots,$$
$$\beta_5 > \beta_6 > \beta_7 > \beta_8 > \cdots,$$
$$\gamma_5 > \gamma_6 > \gamma_7 > \gamma_8 > \cdots.$$

Hence, the word $\overline{W}$ is negatively directed.

Now assume that in the right-hand side of the graphical equality (129) $\alpha_5 < \alpha_6$ and at the same time $\overline{W}$ is not a positively directed word. Then by Lemmas 7.17 and 7.18 there exists a minimal number t such that $\gamma_t < \gamma_{t+1}$ and at the same time $\gamma_{t+1} > \gamma_{t+2}$.

Then, using Lemma 7.17 we get that the prefix

$$\overline{W}_1 \equiv \prod_{i=5}^{t} a^{\alpha_i} b^{\beta_i} c^{\gamma_i}$$

of the word $\overline{W}$ (129) is positively directed and the suffix

$$\overline{W}_2 \equiv \prod_{i=t+2}^{k} a^{\alpha_i} b^{\beta_i} c^{\gamma_i}$$

is negatively directed.

Thus $\overline{W} \equiv \overline{W}_1 a^{\alpha_{t+1}} b^{\beta_{t+1}} c^{\gamma_{t+1}} \overline{W}_2$. This completes the proof of our lemma.

In order to develop the proof of our theorem further we need some generalization of the word of Fibonacci type (see Definition 5.2).

DEFINITION 7.7. Let W (109) be a positively (negatively) directed special word. The word W is called a generalized word of Fibonacci type if in the right-hand side of the graphical equality (109),

$$\alpha_i + \alpha_{i+1} < \alpha_{i+2},\ \beta_i + \beta_{i+1} < \beta_{i+2},\ \gamma_i + \gamma_{i+1} < \gamma_{i+2}$$

(or respectively,

$$\alpha_i > \alpha_{i+1} + \alpha_{i+2},\ \beta_i > \beta_{i+1} + \beta_{i+2},\ \gamma_i > \gamma_{i+1} + \gamma_{i+2}).$$

LEMMA 7.24. *As above, let W (109) be a positively (negatively) directed special word of type 1 with $k > 5$ integer blocks and let $\overline{W}$ (129) be the suffix of W containing $k - 5$ blocs. Then $\overline{W}$ is a generalized word of Fibonacci type.*

Proof. First, let us consider the case when W is a positively directed word. Suppose that $\overline{W}$ is not a generalized word of Fibonacci type. Then for some natural number i, one of the following cases takes place:

(a) $\alpha_{i+2} \leq \alpha_i + \alpha_{i+1}$,

(b) $\beta_{i+2} \leq \beta_i + \beta_{i+1}$,

(c) $\gamma_{i+2} \leq \gamma_i + \gamma_{i+1}$.

Case (a). Since W is a positively directed word, we have in particular that $\alpha_i < \alpha_{i+1} < \alpha_{i+2}$ and by Lemma 7.14,

$$\beta_i < \beta_{i+1}, \ \beta_i < \beta_{i+2}, \ \gamma_i < \gamma_{i+1}. \tag{131}$$

Let us consider the subword

$$\tilde{W} \equiv a^{\alpha_i} b^{\beta_i} c^{\gamma_i} a^{\alpha_{i+1}} b^{\beta_{i+1}} c^{\gamma_{i+1}} a^{\alpha_{i+2}} b^{\beta_{i+2}}$$

of the word $\overline{W}$. By our assumption $\alpha_i \geq \alpha_{i+2} - \alpha_{i+1}$.

Suppose that $\alpha_i > \alpha_{i+2} - \alpha_{i+1}$. This implies that $\tilde{W}$ contains the subword $a^{\alpha_{i+2}-\alpha_{i+1}} b^{\beta_i} c^{\gamma_i} a^{\alpha_{i+1}}$ and at the same time, in view of (131), the word $\tilde{W}$ contains the subword $(c^{\gamma_i} a^{\alpha_{i+2}} b^{\beta_i})$.

Put $u \equiv a^{\alpha_{i+2}-\alpha_{i+1}} b^{\beta_i}$, $v \equiv c^{\gamma_i} a^{\alpha_{i+1}}$. It easy to see that $\tilde{W}$ contains the subword

$$U \equiv auvYvuZ \tag{132}$$

for some $Y, Z \in A^+$. Clearly, in (132) Y is a nonempty word. So, the equality (132) implies that $U = avuYuvZ$ in the semigroup $\mathscr{F}_3$. Since $\tilde{W}$ is a subword of W we get that there exist the words $X, T \in A^+$ such that

$$W = XavuYuvT \equiv V \tag{133}$$

in the semigroup $\mathscr{F}_3$. Since the word v begins with a letter c we obtain that $V \equiv XacT$ for some word T. This contradicts Lemma 7.19. Therefore $\alpha_i = \alpha_{i+2} - \alpha_{i+1}$.

Similar arguments show that in the cases (b) and (c) $\beta_i = \beta_{i+2} - \beta_{i+1}$, $\gamma_i = \gamma_{i+2} - \gamma_{i+1}$. Thus we have

$$W \equiv Xb^{\beta_i} c^{\gamma_i} a^{\alpha_{i+1}} b^{\beta_{i+1}} c^{\gamma_{i+1}} a^{\alpha_i + \alpha_{i+1}} b^{\beta_i + \beta_{i+1}} c^{\gamma_i} Y \ (X, Y \in A^+).$$

Put $u = b^{\beta_i} c^{\gamma_i}$, $v = a^{\alpha_{i+1}} b^{\beta_{i+1}}$. Then $W \equiv Xuvc^{\gamma_{i+1}} a^{\alpha_i} vuY = X(vu)c^{\gamma_{i+1}} a^{\alpha_i}(uv)Y = V$. Clearly, $V \prec W$. This contradicts the fact that the word W is shortlex reduced relative to the semigroup $\mathscr{F}_3$.

The case when W is a negatively directed word can be considered in the same way.

The lemma is proved.

Now we are able to finish the proof of Theorem 7.1.

By Lemma 7.12 the problem of finding upper bound for the growth of the semigroup $\mathscr{F}_3$ can be reduced to the same problem for the set of all integer perfect words of type 1. Using Lemmas 7.23 and 7.24 we obtain that the last problem can be reduced to the problem of finding the upper bound of the growth for the set of all positively (negatively) directed generalized words of Fibonacci type.

Let F be a set of all generalized words of Fibonacci type. Clearly, every word W (109) belonging to F is in obvious one-to-one correspondence with an ordered triple $(\{\alpha_n\},\{\beta_n\},\{\gamma_n\})$ of the number sequences. Since these sequences determine the words

$$\prod_{i=1}^{k} a^{\alpha_i} b^{\alpha_i} c^{\alpha_i}, \ \prod_{i=1}^{k} a^{\beta_i} b^{\beta_i} c^{\beta_i}, \ \prod_{i=1}^{k} a^{\gamma_i} b^{\gamma_i} c^{\gamma_i}$$

of Fibonacci type, we obtain that the growth $[\mathbf{g}_F]$ of the set F is no larger than the cube of the growth of the set of all words of Fibonacci type. Therefore, by Theorem 5.1 we obtain that

$$[\mathbf{g}_F] \prec [\mathbf{m}^{\mathbf{3\cdot 2\cdot 1}\ln \mathbf{m}}] = [\mathbf{m}^{\mathbf{6\cdot 3}\ln \mathbf{m}}]. \tag{134}$$

As above, let $\mathscr{Q}$ and $\mathscr{P}_1$ be the sets of all special words and perfect words of type 1 respectively.

Combining Lemma 7.23 and relation (134) we obtain that the growth $[\mathbf{g}_{\mathscr{Q}}]$ of the set $\mathscr{Q}$ satisfies the inequality

$$[\mathbf{g}_{\mathscr{Q}}] \prec [\mathbf{m}^{\mathbf{2\cdot 6\cdot 3}\ln \mathbf{m}}] = [\mathbf{m}^{\mathbf{12\cdot 6}\ln \mathbf{m}}].$$

Then, taking into account Corollary 7.3, from Lemma 7.22 we have that the growth $[\mathbf{g}_{\mathscr{P}_1}]$ of the set $\mathscr{P}_1$ satisfies the inequality

$$[\mathbf{g}_{\mathbf{P}_1}] \prec [\mathbf{m}^{\mathbf{3\cdot 12\cdot 6}\ln \mathbf{m}}] = [\mathbf{m}^{\mathbf{37\cdot 8}\ln \mathbf{m}}].$$

Now, by combining this result with Lemma 7.12, we obtain $[\mathbf{g}_{\mathscr{F}_3}] \prec [\mathbf{m}^{\mathbf{37\cdot 8}\ln \mathbf{m}}]$.

Theorem 7.1 is proved.

8. GROWTH AND GELFAND–KIRILLOV SUPERDIMENSION IN SEMIGROUP VARIETIES

In this section we shall obtain some new results about the asymptotic behavior of a semigroup satisfying a given identity. Our goal is to give examples of nonperiodic and periodic semigroup varieties containing semigroups with arbitrarily small intermediate growth.

As in Section 6, let $\mathbf{B}$ be the variety of all nilpotent (in the sense of Neumann and Taylor) semigroups of class 2 which can be defined by the identity (84).

THEOREM 8.1. *Let $\psi(m)$ be an arbitrary monotone-nondecreasing function from $\mathbf{N}$ into $\mathbf{R}^+$ such that*

$$\psi(m) < \ln m \qquad (m > 1) \tag{135}$$

and

$$\lim_{m \to \infty} \psi(m) = \infty. \tag{136}$$

Then there exists a 3-*generated semigroup* $\Pi \in \mathbf{B}$ *such that*

$$[\mathbf{e}^{(1/9)\psi(\mathbf{m})\ln \mathbf{m}}] \prec [\mathbf{g}_\Pi] \preceq [\mathbf{e}^{\psi(\mathbf{m})\ln \mathbf{m}}]. \tag{137}$$

Proof. Put

$$\theta(m) = e^{(1/2)\psi(m)} \tag{138}$$

and

$$r = \left[\log_4 \frac{m}{\theta(m)}\right]. \tag{139}$$

It follows from (135) and (138) that

$$\theta(m) < \sqrt{m} \qquad (\text{for } m = 2, 3, \dots). \tag{140}$$

Thus if m is sufficiently large,

$$r > \frac{1}{4} \ln m. \tag{141}$$

We also mention that in view of (136)

$$\lim_{m \to \infty} \theta(m) = \infty.$$

Define a sequence

$$\mathscr{M}_{1,m}, \mathscr{M}_{2,m}, \dots, \mathscr{M}_{r,m}$$

of segments of the linearly ordered set $< \mathbf{N}, \leq, >$ by the rule

$$\mathscr{M}_{i,m} = \{n \in \mathbf{N} : 3^{i-1}[\theta(m)] < n \leq (3^{i-1} + 1)[\theta(m)]\}. \tag{142}$$

LEMMA 8.1. *Let* $m > 1$ *be a natural number and* r *be defined by the equality* (139). *Also, let*

$$\alpha_1, \alpha_2, \dots, \alpha_r \tag{143}$$

be a finite sequence of natural numbers such that

$$\alpha_i \in \mathscr{M}_{i,m} \qquad (i = 1, 2, \dots, r). \tag{144}$$

Then

(a) *the sequence* $\{\alpha_n\}$ *satisfies the inequalities* (68) *and* (69) *from the definition of the positively directed word of Fibonacci type.*

(b) *if* $r > 3$ *then*

$$\sum_{i=1}^{r} \alpha_i < \frac{m}{3}, \tag{145}$$

and

(c) $$2\alpha_i < \alpha_{i+1} < 4\alpha_i \qquad (i = 2, \dots, r-1). \tag{146}$$

In particular,

$$\sum_{i=1}^{q} \alpha_i < \alpha_{q+1} < 3 \sum_{i=1}^{q} \alpha_i. \tag{147}$$

Proof. It follows from (142), (144) that

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_r$$

and

$$\alpha_i \le (3^{i-1} + 1)\theta(m), \; \alpha_{i+1} \le (3^i + 1)\theta(m), \; \alpha_{i+2} > 3^{i+1}\theta(m).$$

Thus

$$\alpha_{i+2} > \alpha_i + \alpha_{i+1}$$

and we obtain that the condition (a) holds.

The same arguments show that

$$\sum_{i=1}^{r} \alpha_i < 2\theta(m) \sum_{i=1}^{r} 3^{i-1} = \theta(m)(3^r - 1).$$

Now suppose that $r > 3$. Then $(3^r - 1) < 4^r/3$. Therefore

$$\sum_{i=1}^{r} \alpha_i < \frac{\theta(m)4^r}{3} < \frac{\theta(m)4^{\log_4(m/\theta(m))}}{3} = \frac{m}{3}.$$

Thus, the condition (b) holds.

The inequalities (146) and (147) can be checked in an analogous way.

This completes the proof of the lemma.

Let us fix a natural number m and let $\mathscr{H}_m$ be the set of all possible finite sequences $\{\alpha_n\}$ (143) satisfying condition (144). Since

$$\text{card } \mathscr{M}_{i,1} = \text{card } \mathscr{M}_{i,2} = \cdots = \text{card } \mathscr{M}_{i,r} = [\theta(m)],$$

we have that

$$\text{card } \mathscr{H}_m = \prod_{i=1}^{r} \text{card } \mathscr{M}_{i,m} = \left(\text{card}[\theta(m)]\right)^r = [e^{1/2\psi(m)}]^{[\log_4(m/\theta(m))]}. \tag{148}$$

Clearly

$$\left[\log_4 \frac{m}{\theta(m)}\right] < \log_4 m = \frac{\ln m}{\ln 4}.$$

Thus, (148) shows that

$$\operatorname{card} \mathscr{H}_m < e^{\psi(m)\ln m} \tag{149}$$

for sufficiently large numbers m.

On the other hand, if m is sufficiently large, by combining (148) and (141) we obtain

$$\begin{aligned}\operatorname{card} \mathscr{H}_m &> \left(e^{(1/2)\psi(m)} - 1\right)^{(\log_4(m/\theta(m))-1)} > \left(e^{(1/2\psi(m)-1)(((1/4)\ln m)-1)}\right)\\ &> e^{(1/9)\psi(m)\ln m}.\end{aligned} \tag{150}$$

For any sequence $\{\alpha_n\} \in H_m$ put

$$W_{\{\alpha_n\}} \equiv a^{\alpha_1} b^{\alpha_1} c^{\alpha_1} a^{\alpha_2} b^{\alpha_2} c^{\alpha_2} \cdots a^{\alpha_r} b^{\alpha_r} c^{\alpha_r}.$$

We say that $W_{\{\alpha_n\}}$ is an *integer word corresponding to the sequence* $\{\alpha_n\}$.

As in Sections 6 and 7, let $\mathscr{F}_3$ be a **B**-free semigroup over the alphabet $\{a, b, c\}$.

It follows from Lemma 8.1 that $W_{\{\alpha_n\}}$ is a positively directed word of Fibonacci type. Then by Lemma 6.3 $W_{\{\alpha_n\}}$ is an isoterm relative to the semigroup $\mathscr{F}_3$. We note also that by Lemma 8.1, if $r > 3$ then

$$l\left(W_{\{\alpha_n\}}\right) < 3\sum_{i=1}^{r} \alpha_i < 3 \cdot \frac{m}{3} = m.$$

Let $\mathscr{H} = \bigcup_{m=1}^{\infty} \mathscr{H}_m$ and let $\tau\colon \mathscr{H} \to \mathscr{F}_3$ be a map such that

$$\tau(\{\alpha_n\}) = W_{\{\alpha_n\}}.$$

Since $W_{\{\alpha_n\}}$ is an isoterm relative to $\mathscr{F}_3(\mathbf{B})$ we have that τ is an injective map. In particular,

$$\operatorname{card} \mathscr{H}_m = \operatorname{card}(\mathscr{H}_m)_\tau. \tag{151}$$

Put $\mathscr{K}_m = (\mathscr{H}_m)\tau$ and $\mathscr{K} = \bigcup_{i=1}^{m} \mathscr{K}_m$. Let $\tilde{\mathscr{K}}_m$ be the set of all subwords of the set $\mathscr{K}_m$ and $\tilde{\mathscr{K}} = \bigcup_{i=1}^{m} \tilde{\mathscr{K}}_m$.

Let $\mathscr{J} = \mathscr{F}_3 \setminus \tilde{\mathscr{K}}$ be an ideal of the semigroup $\mathscr{F}_3$ and $\Pi = \mathscr{F}_3/\mathscr{J}$ be a Rees-factor semigroup with respect to the ideal $\mathscr{J}$. As we have already noted, every element of $\mathscr{K}$ is an isoterm relative to the semigroup $\mathscr{F}_3$. Therefore, the semigroup Π belongs to the variety **B**.

The statement of our theorem will follow from the following lemma.

LEMMA 8.2. *The growth* $[\mathbf{g}_\Pi]$ *of the semigroup* Π *is greater than the growth of the function* $e^{(1/9)\psi(m)\ln m}$ *and less than the growth of the function* $e^{\psi(m)\ln m}$.

Proof. Let U be a word over the alphabet $\{a, b, c\}$ having height ≥ 7. Let $l(U) = d$ and let us also assume that $U \neq 0$ in a semigroup Π. Then U is a subword of some word $W_{\{\alpha_n\}}$ from the set $\mathscr{H}_m$ for some natural number m. Let

$$V \equiv a^{\alpha_1} b^{\alpha_1} c^{\alpha_1} a^{\alpha_2} b^{\alpha_2} c^{\alpha_2} \cdots a^{\alpha_q} b^{\alpha_q} c^{\alpha_q}$$

be a minimal initial integer block of the word $W_{\{\alpha_n\}}$ containing U as a subword. In particular,

$$l(U) \leq l(V) = 3 \sum_{i=1}^{q} \alpha_i.$$

Since $h(U) \geq 7$, a word $a^{\alpha_{q-1}} b^{\alpha_{q-1}} c^{\alpha_{q-1}}$ must be a subword of U. In particular, $\alpha_{q-1} \leq \frac{1}{3} l(U) = \frac{1}{3} d$. Then, by Lemma 8.1 we obtain that $\sum_{i=1}^{q-2} \alpha_i < \frac{1}{3} d$. This shows that

$$\sum_{i=1}^{q-1} \alpha_i < \frac{2}{3} d.$$

By virtue of (146) we obtain $\alpha_q < 4\alpha_{q-1} < \frac{4}{3} d$. Therefore

$$\sum_{i=1}^{q} \alpha_q < \frac{2}{3} d + \frac{4}{3} d = 2d.$$

This means that $l(V) < 6d$. Thus $U \in \tilde{\mathscr{K}}_{6d}$ and we get the following upper bound for the growth function $g_\Pi(m)$ of a semigroup Π:

$$g_\Pi(m) < \operatorname{card} \tilde{\mathscr{K}}_{6m}. \tag{152}$$

Now let $f(m) = \operatorname{card} \tilde{\mathscr{K}}_m$. Clearly, in order to complete our proof, by virtue of the inequality (152) it suffices to show that

$$[\mathbf{f}(\mathbf{m})] \preceq [\mathbf{e}^{\psi(\mathbf{m}) \ln \mathbf{m}}]. \tag{153}$$

As above, let $\{\alpha_n\}$ be a finite sequence of natural numbers (143) that has been considered in Lemma 8.1. In particular, this sequence satisfies the condition (145).

Also, let $\overline{W}$ be a subword of the word $W_{\{\alpha_n\}}$ with height ≥ 5. It is easy to see that if the sequence $\{\alpha_n\}$ is fixed, $\overline{W}$ determines uniquely by its initial and terminal segments of height 3. At the same time, each such segment is determined by its length (which is always less m) and the number of corresponding blocks (which is always $\leq r$). Therefore, for every fixed number m, the total number of different subwords of the word $W_{\{\alpha_n\}}$ is strongly less than $(m^3 r)^2 = m^6 r^2$.

Thus, $f(m) = \text{card}\ \tilde{\mathscr{K}}_m < r^2 m^6 \text{card}\ \mathscr{K}_m$. It follows from (151) that card $\mathscr{K}_m$ = card $\mathscr{H}_m$. Now, using (139) and (149) we obtain

$$f(m) < \left[\log_4 \frac{m}{\theta(m)}\right]^2 m^6 e^{\psi(m)\ln m}. \tag{154}$$

This shows that relation (153) holds.

Since different elements of the set $\tilde{\mathscr{K}}_m$ correspond to different elements of the semigroup Π, the inequality (150) shows that $g_\Pi(m) > m^{(1/9)\psi(m)}$. So,

$$[\mathbf{e}^{(1/9)\psi(\mathbf{m})\ln \mathbf{m}}] \prec [\mathbf{g}_{\mathbf{\Pi}}].$$

In particular, the growth of Π is greater than polynomial. This completes the proof of the lemma.

Theorem 8.1 is proved.

COROLLARY 8.1. *There exists a continuum of different types of growth for the semigroups with three generators belonging to the variety* **B** *and having zero Gelfand–Kirillov superdimension.*

Proof. For any real positive number ξ $(0 < \xi < 1)$ put $\psi_\xi(m) = (\ln m)^\xi$. Clearly,

$$\psi_\xi(m) < \ln m \qquad (m > 2)$$

and

$$\lim_{m\to\infty} \psi_\xi(m) = \infty.$$

Thus by Theorem 8.1 there exists a semigroup $\Pi_\xi \in \mathbf{B}$ such that

$$\left[m^{(1/9)(\ln m)\xi+1}\right] \prec [\mathbf{g}_{\Pi_\xi}] \prec \left[m^{(\ln m)^{\xi+1}}\right]. \tag{155}$$

It is easy to see that if ξ is as above and $\eta \in (0,1)$ then in view of (155) the inequality $\xi < \eta$ implies that the growth $[\mathbf{g}_{\mathbf{\Pi}_\xi}]$ of a semigroup Π_ξ is strongly less than the growth $[\mathbf{g}_{\Pi_\eta}]$ of a semigroup Π_η.

Note that the variety **B** contains an infinite monogenic semigroup and therefore this variety is nonperiodic. Combining the classical results of Morse and Hedlund [19] about the so-called square-free and cube-free infinite sequences with our methods, we find a periodic variety with the properties described in the formulation of Theorem 8.1.

THEOREM 8.2. *Let $\psi(m)$ be a function satisfying the conditions of Theorem* 8.1. *Then there exists a* 4*-generated semigroup* Π_1 *with the identity*

$$x^3 = 0 \tag{156}$$

such that

$$[\mathbf{e}^{(1/9)\psi(\mathbf{m})\ln \mathbf{m}}] \prec [\mathbf{g}_{\Pi_1}] \preceq [\mathbf{e}^{\psi(\mathbf{m})\ln \mathbf{m}}]. \tag{157}$$

Proof. Let F_2 be the free semigroup over the alphabet $\{a, b\}$, let φ: $F_2 \to F_2$ be an endomorphism such that $\varphi(a) \equiv ab$ and $\varphi(b) \equiv ba$, and let

$$a, \varphi(a), \varphi^2(a), \dots, \varphi^n(a), \dots$$

be the Morse–Hedlund sequence [19]. This sequence has the form

$$ab, abba, abbabaab, \dots,$$

and it defines the hyperword

$$\Delta = abbabaabbaababba\dots$$

which does not contain subwords of type E^3.

Let n be a natural number and $\Delta^{(n)}$ the initial segment of Δ having length n. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a finite number sequence determining the positively directed word of Fibonacci type.

Clearly, we can represent a word $\Delta^{2(\alpha_1+\alpha_2+\cdots+\alpha_n)}$ in the form

$$\Delta^{2(\alpha_1+\alpha_2+\cdots+\alpha_n)} \equiv E_1T_1E_2T_2 \cdots E_nT_n \tag{158}$$

where E_i, T_i are the words over the alphabet $\{a, b\}$ such that

$$l(E_i) = l(T_i) = \alpha_i \qquad (i = 1, 2, \dots, n). \tag{159}$$

Let $\Gamma = \{a, b\}$, $\Sigma = \{u, v\}$, and Γ^+, Σ^+ be free semigroups over alphabets Γ and Σ, respectively. Let

$$\phi\colon \Gamma^+ \to \Sigma^+$$

be an isomorphism such that

$$\phi(a) = u, \qquad \phi(b) = v.$$

Put

$$\Phi = \Gamma \cup \Sigma = \{a, b, u, v\},$$

and as above let $E_1, T_1, E_2, T_2, \dots, E_n, T_n$ be the words uniquely determined by the conditions (158) and (159). Let

$$\Delta^*_{(\alpha_1, \alpha_2, \dots, \alpha_n)} \equiv E_1(T_1\phi)E_2(T_2\phi) \cdots E_n(T_n\phi). \tag{160}$$

The equality (160) shows that ϕ induces a one-to-one correspondence between the set of all finite sequences $\{\alpha_n\}$ as above and the set of all words of the type $\Delta^*_{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ over the alphabet Φ. Since Δ does not contain a subword of the type E^3 and ϕ is an isomorphism, the word $\Delta^*_{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ is also cube free.

We shall use the main ideas and the main constructions given in the process of the proof of Theorem 8.1. In particular, we refer the reader to the definition of the function $\theta(m)$ (138), the number r from the equality (139), and the sequence of segments $\mathscr{M}_{i,m}$ (142). Let $\mathscr{H}_m$ and $\mathscr{H}$ be the sets of a finite number of sequences of a special type introduced in the proof of Theorem 8.1 and $W_{\{\alpha_n]}$ be an integer word corresponding to the sequence $\{\alpha_n\}$ (143).

For any sequence $\{\alpha_n\} \in \mathscr{H}_m$ put

$$W^*_{\{\alpha_n\}} \equiv \Delta^*_{(\alpha_1, \alpha_2, \dots, \alpha_n)}.$$

Clearly, $l(W^*_{\{\alpha_n\}}) = 2\sum_{i=1}^n \alpha_i$. Then, by Lemma 8.1,

$$l\left(W^*_{\{\alpha_n\}}\right) < \frac{2}{3}m.$$

Let Ł be the variety of semigroups defined by the identity $X^3 = 0$ and $\mathscr{F}_4$(Ł) be a free semigroup over 4 letters of the alphabet Φ with respect to Ł. Let $\tau^*\colon \mathscr{H} \to \mathscr{F}_4$(Ł) be a map such that $\tau^*(\{\alpha_n\}) = W^*_{\{\alpha_n\}}$. Since the word $\Delta^*_{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ is cube-free, we have that $W^*_{\{\alpha_n\}}$ is an isoterm relative to the semigroup $\mathscr{F}_4$(Ł), τ^* is an injective map, and (compare with the equality (151)) card $\mathscr{H}_m = \operatorname{card}(\mathscr{H}_m)\tau^*$.

Let $\mathscr{Q}$ be the set of all subwords of the words of type $W^*_{\{\alpha_n\}}$ and $J^* = \mathscr{F}_4(\text{Ł}) \setminus \mathscr{Q}$. Also, let $\Pi_1 = \mathscr{F}_4(\text{Ł})/J^*$ be the Rees-factor semigroup with respect to the ideal J^*. Clearly, Π_1 belongs to the variety Ł. Now we are going to show that the relations (157) hold.

Indeed, let us fix a natural number m and let U be a word over the alphabet Φ, such that $l(U) = m$ and $l(U) \neq 0$ in the semigroup Π_1. Then U is a subword of some word $\Delta^*_{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ (160). Therefore, we may represent U in one of the following forms:

(1) $U \equiv E'_i(T_i\phi)\overline{U}E'_{j+1}$,

(2) $U \equiv (T_i\phi)'\overline{U}E'_{j+1}$,

(3) $U \equiv E'_i T_i \varphi \overline{U} E_{j+1}(T_j\phi)'$,

(4) $U \equiv (T_i\phi)'\overline{U}E_{j+1}(T_j\phi)'$,

where $\overline{U} \equiv E_{i+1}(T_{i+1}\phi)E_{i+2}(T_{i+2}\phi)\cdots E_j(T_j\phi)$, E'_{j+1} and $(T_j\phi)'$ are the initial segments of the words E_{j+1} and T_j, respectively, and the words E'_i and $(T_i\phi)'$ are the terminal segments of the words E_i and T_j, respectively.

Obviously, the word $\overline{U}$ corresponds to a number sequence $\alpha_{i+1} < \alpha_{i+2} < \cdots < \alpha_j$ determining the positively directed word of Fibonacci type and such that

$$\alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_j < m.$$

Let $h(m)$ be the number of all the different subwords of the hyperword Δ having length m. It follows from [19] that for any natural number $m > 1$,

$$h(m) < 6m. \tag{161}$$

Let us fix the sequence $\{\alpha_n\}$. Since ϕ is an isomorphism and the words $E_i, E_j, T_i\phi, T_j\phi$ are subwords of the hyperword Δ with length $\leq m$, we obtain that for each fixed choice of the word $\overline{U}$ we have $< (6(m))^2$ (that is, $< 36m^2$) different possibilities for the word U. Repeating the arguments from Lemma 8.2 (see the proof of the formula (154)) we obtain that each fixation of the subword $\overline{U}$ gives us $4r^2 \cdot 36m^2$ $(= 144r^2m^2)$ or fewer different possibilities for U.

This shows that if we do not fix the sequence $\{\alpha_n\}$ then (compare this with the inequality (154)) the word U may be chosen fewer than

$$144r^2m^2 \operatorname{card} \mathscr{H}_m = 144\left[\log_4 \frac{m}{\theta(m)}\right]^2 m^6 e^{\psi(m)\ln m}$$

different ways. Therefore, the growth function $g_{\Pi_1}(m)$ of the semigroup Π_1 satisfies the inequality

$$\begin{aligned} g_{\Pi_1}(m) &< m\left(144\left[\log_4 \frac{m}{\theta(m)}\right]^2 m^6 e^{\psi(m)\ln m}\right) \\ &= 144\left[\log_4 \frac{m}{\theta(m)}\right]^2 m^7 e^{\psi(m)\ln m}. \end{aligned}$$

This means that

$$[\mathbf{g}_{\Pi_1}] \preceq [\mathbf{e}^{\psi(\mathbf{m})\ln \mathbf{m}}].$$

On the other hand, clearly each choice of the sequence $\{\alpha_n\} \in H_m$ determines a unique word $W^*_{\{\alpha_n\}}$. Thus $\operatorname{card}\mathscr{H}_m \preceq [\mathbf{g}_{\Pi_1}]$, and taking into account (150) we obtain the relations (157).

The theorem is proved.

The proof of the following statement is based on our method and the construction (using the word Δ) of an infinite cube-free word over a three-letter alphabet (see [22]). We also use the fact that this hyperword contains fewer than $12m$ different subwords of length m (compare this with the inequality (161)).

THEOREM 8.3. *Let* $\psi(m)$ *be a function satisfying the conditions of Theorems* 8.1 *and* 8.2. *Then there exists a* 6-*generated semigroup* Π_2 *with the identity*

$$x^2 = 0$$

such that $[\mathbf{e}^{\mathbf{1/9}\psi(\mathbf{m})\ln \mathbf{m}}] \prec [\mathbf{g}_{\mathbf{\Pi}_2}] \preceq [\mathbf{e}^{\psi(\mathbf{m})\ln \mathbf{m}}]$.

COROLLARY 8.2. *There exists a continuum of different types of growth for the f.g. semigroups belonging to the variety generated by the identity* $x^2 = 0$ *and having zero Gelfand–Kirillov superdimension.*

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